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A spectral problem with an indefinite weight for an elliptic system *

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Abstract

We establish the completeness and the summability in the sense of Abel-Lidskij of the root vectors of a non-selfadjoint elliptic problem with an indefinite weight matrix and the angular distribution of its eigenvalues.

1 Introduction

Spectral problems with an indefinite weight for typically non-selfadjoint elliptic problems (not obtained by perturbation of selfadjoint ones) have been initially investigated by (Faierman 1990). He obtained the completeness of the root vectors and the angular distribution of eigenvalues for a regular scalar elliptic problem. We refer to his paper for further references on this topic. Similar problems for non-selfadjoint elliptic systems have not yet been investigated. The present paper aims to initialize such investigations. In a bounded region Ω in $\mathbf{R}^n$ with a $(n-1)$–dimensional boundary Γ, we consider the boundary value problem with an indefinite weight matrix

$$(A-\lambda E_\omega)\,u=0 \text{ in } \Omega;\quad B_k u=0\ (k=1,\dots,r) \text{ on } \Gamma, \tag{1}$$

where, A is a square matrix of dimension N consisting of differential operators of order $2m$ with complex coefficients, B_k $(k=1,\dots,r=Nm)$ are $N-$ dimensional rows whose components are differential operators of order $m_k \le 2m-1$ with complex coefficients and E_ω is a diagonal matrix whose diagonal entries are real-valued functions.

In this work, we establish some results on the completeness and the summability by Abel's method (see e.g., Lidskij 1962, Agranovish 1977, Kostyuchenko-Razdievskij 1974, for details) of the root vectors of problem (1), and the angular distribution of its eigenvalues. In order to derive our results, we establish the unique solvability of an auxilliary elliptic transmission problem with a parameter. We note that our smoothness assumptions on problem (1) are slightly

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weaker than in (Faierman 1990), since we do not use the formally adjoint problem to (1). This approach can be successfully used to investigate similar problems for elliptic systems in the sense of Douglis-Nirenberg. We refer to (Agranovish 1990, Kozhevnikov 1973), where such problems have been investigated in the case when E_ω is the unit $N \times N$ matrix. We also note the important earlier contributions of (Agmon 1962) and (Grisvard and Geymonat 1967). The work is organized as follows. Section 2 is devoted to the formulation of the main assumptions. In section 3, we establish the unique solvability of an elliptic transmission problem with a parameter. In section 4, we derive our main results.

2 Basic assumptions

Let $x = (x_1, \dots, x_n)$, $D_j = -i\frac{\partial}{\partial x_j}$, $D = (D_1, \dots, D_n)$, $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$, where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{Z}_+^n$ and $|\alpha| = \sum_{j=1}^n \alpha_j$,

$$\begin{aligned} A &= A(x, D) = \left\{ \sum_{|\alpha| \le 2m} a_\alpha^{ij}(x) D^\alpha \right\}_{i,j=1}^N, \\ B_k &= B_k(x, D) = \left\{ \sum_{|\beta| \le m_k} b_\beta^{kj}(x) D^\beta \right\}_{j=1,\dots,N} \qquad (k = 1, \dots, Nm). \end{aligned}$$

We assume throughout that the operators A, B_k $(k = 1, \dots, Nm)$ and the domain Ω satisfy the following smoothness conditions: the region Ω is of class $\mathbf{C}^{2m}$, the coefficients $a_\alpha^{ij}(x)$ in A are continuous in $\bar{\Omega}$, for $|\alpha| = 2m$ and bounded for $|\alpha| \le 2m-1$, the coefficients $b_\beta^{kj}(x)$ in B_k belong to $\mathbf{C}^{2m-m_k}(\Gamma)$ for $|\beta| = m_k$ and bounded together with their derivatives of order up to $2m - m_k$ for $|\beta| \le m_k - 1$. Here $\mathbf{C}^l$ denotes the space of all functions continuous together with their derivatives of order up to l.

Let $H_l(\Omega, N)$ (l is an integer) be the direct product of N Sobolev spaces $W_2^l(\Omega)$. When $l = 0$, we write $H_0(\Omega, N) = L_2(\Omega, N)$. We denote by $H_{l-\frac{1}{2}}(\Gamma)$ $(l \ge 1)$ the space of boundary values of functions from $W_2^l(\Omega)$ and by $H_{l-\frac{1}{2}}(\Gamma, N)$ the direct product of N such spaces. The norms in $H_l(\Omega, N)$ and $H_{l-\frac{1}{2}}(\Gamma)$ are respectively denoted by $\|\ \|_{l,\Omega}$ and $\|\ \|_{l-\frac{1}{2},\Gamma}$.

Next we turn to the assumptions concerning the weight matrix $E_\omega(x)$. They will be closely related to a certain partition of the domain Ω into appropriate subdomains.

Assumption 1. Let there be given some $(n-1)$–dimensional manifolds $\Gamma_1, \dots, \Gamma_q$ each of class C^{2m}, lying inside Ω, having no point in common with Γ and such that $\Gamma_l \cap \Gamma_k = \emptyset$ for $l \neq k$. They divide Ω into subdomains

$$\Omega_1, \dots, \Omega_{q+1}.$$

We assume that the diagonal entries $\omega_s(x)$ $(s = 1, \dots, N)$ of $E_\omega(x)$ are continuous in each $\bar{\Omega}_l$, can pertain a discontinuity of first kind and change sign while crossing any Γ_p and

$$|\omega_s(x)| > 0 \; (s = 1, \dots, N).$$

Since the functions $\omega_s(x)$ $(s = 1, \dots, N)$ are assumed to be discontinuous across Γ_p, the solution of (1) may not belong to the functional space $H_{2m}(\Omega, N)$ which is of interest to us. Thus, in order to preserve the membership of the solution to this space, we impose, among others, the following natural conjugation conditions:

$$D_n^j u_{(l)}(x) = D_n^j u_{(l')}(x)\,; (j = 0, \dots, 2m-1) \text{ on each } \Gamma_p, \tag{2}$$

where $u_{(l)}$ and $u_{(l')}$ are the restrictions of the function u to Ω_l and $\Omega_{l'}$ respectively, Γ_p separates Ω_l from $\Omega_{l'}$ and D_n is the derivative along the inward normal to Γ_p.

Definition 1 A complex number λ will be called an eigenvalue of the boundary problem (1) if the problem (1) with the transmission conditions (2) admits at least a non-trivial solution $u \in H_{2m}(\Omega, N)$; this solution is refered to as the eigenfunction of (1) corresponding to λ; otherwise the number λ is called regular point of (1) .

Next, let $A_0(x, \xi)$ and $B_{0k}(x, \xi)$ ($k = 1, \dots, Nm$) be respectively the principal parts of the operators A and B_k ($k = 1, \dots, Nm$). Let also

$$\Xi(\theta_1, \theta_2) = \{\lambda : \; \theta_1 \leq \arg \lambda \leq \theta_2\}$$

be an angular sector in the complex plane and λ an element of $\Xi(\theta_1, \theta_2)$.

Assumptions 2.

(i) We require the matrix

$$A_0(x, \xi) - \lambda E_\omega(x) \tag{3}$$

to be invertible for all $\xi \in R^n$; $|\xi| + |\lambda| \neq 0$ and all $x \in \bar{\Omega}_p$ $(p = 1, \dots, q+1\ )$.

(ii) Let x_0 be any point on Γ. We shall turn the coordinates axes such that, the axis x_n takes the direction of the inward normal to Γ at x_0. For simplicity, we suppose that the operators A, B_k are written in the system of coordinates connected with x_0. We consider the following problem on the ray.

$$(A_0(x_0, \xi', D_t) - \lambda E_\omega(x_0))v(t) = 0, \qquad t > 0 \tag{4}$$

$$B_{0k}(x_0,\xi',D_t)v(t)|_{t=0}=g_k \quad (k=1,\dots,Nm), \tag{5}$$

where $\xi'=(\xi_1,\dots,\xi_{n-1})$ and $D_t=-i\frac{d}{dt}$.

For any $\xi'\in R^{n-1}$, $|\xi'|+|\lambda|\neq 0$, the space of solutions of system (4), exponentially decreasing in modulus when $t\to\infty$, is Nm -dimensional and the problem (4)-(5) is uniquely solvable for any g_k , in this space.

(iii) Let A_l ($A_{l'}$) and $E_{\omega l}$ ($E_{\omega l'}$) be respectively the restriction of the matrix A and the matrix E_ω to Ω_l ($\Omega_{l'}$) and assume that Γ_p separates Ω_l and $\Omega_{l'}$. We take a point $x_0\in\Gamma_p$ and turn the coordinate axes such that, the axis x_n takes the direction of the inward normal to Γ_p at x_0. We consider the following transmission problem on the line,

$$(A_{l0}(x_0,\xi',D_t)-\lambda E_{\omega l}(x_0))\,v_l(t)=0\,,\quad t>0 \tag{6}$$

$$(A_{l'0}(x_0,\xi',D_t)-\lambda E_{\omega l'}(x_0))\,v_{l'}(t)=0,\quad t<0 \tag{7}$$

$$D_t^\mu v_l(0)-D_t^\mu v_{l'}(0)=h_{\mu p}\,,\quad \mu=0,\dots,2m-1. \tag{8}$$

For any $\xi'\in R^{n-1}$; $|\xi'|+|\lambda|\neq 0$, the space of solutions of the system (6)-(7), exponentially decreasing in modulus when $|t|\to\infty$, is $2Nm$–dimensional and the problem (6)-(8) is uniquely solvable in this space for any
N-dimensional column $h_{\mu p}$.

We define an exponentially decreasing solution of (6)-(7) as a vector $v=(v_l,v_{l'})$, where v_l and $v_{l'}$ are respectively solutions of (6) and (7) and $v_l\,(t)\to 0$ when $t\to+\infty$ while $v_{l'}\,(t)\to 0$ when $t\to-\infty$.

3 An auxilliary result

In this section we establish an auxilliary result on the unique solvability of an elliptic transmission problem with a parameter. This result plays a central role in our investigations.

Let us consider the following non-homogeneous transmission problem induced by the boundary value problem (1) and the conjugation conditions (2):

$$L_l u\,(x)=(A_l\,(x,D)-\lambda E_{\omega l}\,(x))\,u_l\,(x)=f_l\,(x)\ \text{ in } \Omega_l\ (l=1,\dots,q+1), \tag{9}$$

$$[D_n^\mu u]_p=D_n^\mu u_l\,(x)-D_n^\mu u_{l'}\,(x)=h_{\mu p}\,(x)\ \text{ on } \Gamma_p\ (\mu=0,\dots,2m-1;\ p=1,\dots,q), \tag{10}$$

$$B_k\,(x,D)\,u\,(x)=g_k\,(x)\ \text{ on } \Gamma\ (k=1,\dots,Nm), \tag{11}$$

where A_l, $E_{\omega l}$ and Γ_p have the same meaning as in Assumption 2(iii), f_l and $h_{\mu p}$ are vector-columns of height N defined respectively in Ω_l and Γ_p, g_k are scalar functions defined on Γ.

Now let $\mathcal{U}$ be the operator connected with the problem (9)-(11), defined by

$$\begin{aligned}\mathcal{U}u &= \Big\{L_1u_1,\dots,L_{q+1}u_{q+1},[u]_1,\dots,\left[D_n^{2m-1}u\right]_1,\dots,[u]_q,\dots,\left[D_n^{2m-1}u\right]_q, \\ &\quad B_1u,\dots,B_{Nm}u\}\end{aligned} \tag{12}$$

and acting from

$$\mathcal{H}\left(\Omega^{(l)},N\right)=H_{2m}\left(\Omega_1,N\right)\times\cdots\times H_{2m}\left(\Omega_{q+1},N\right) \tag{13}$$

to

$$\begin{aligned}&\mathcal{H}\left(\Omega,\Gamma^{(p)},\Gamma\right) \\ &= \prod_{p=1}^{q+1}L_2\left(\Omega_l,N\right)\times\prod_{p=1}^{q}\prod_{\mu=0}^{2m-1}H_{2m-\mu-\frac12}\left(\Gamma_p,N\right)\times\prod_{k=1}^{Nm}H_{2m-m_k-\frac12}\left(\Gamma\right).\end{aligned} \tag{14}$$

We introduce in $H_l\left(\Omega,N\right)$ and $H_{l-\frac12}\left(\Gamma\right)$ respectively the norms

$$\begin{aligned}|||\ |||_{l,\Omega} &= ||\ ||_{l,\Omega}+|\lambda|^{\frac{l}{2m}}\,||\ ||_{0,\Omega}\,, \\ |||\ |||_{l-\frac12,\Gamma} &= ||\ ||_{l-\frac12,\Gamma}+|\lambda|^{\frac{l-\frac12}{2m}}\,||\ ||_{0,\Gamma}\,.\end{aligned}$$

Theorem 1 *Let Assumptions 1 and 2 be satisfied. Then for sufficiently large $\lambda\in\Xi\left(\theta_1,\theta_2\right)$, for each $f_l\in L_2\left(\Omega_l,N\right)$, $h_{\mu p}\in H_{2m-\mu-\frac12}\left(\Gamma_p,N\right)$ and $g_k\in H_{2m-m_k-\frac12}\left(\Gamma\right)$, the problem (9)-(11) has a unique solution $u\in\mathcal{H}\left(\Omega^{(l)},N\right)$ which satisfies the estimate*

$$C^{-1}\,|||u|||_{\mathcal{H}\left(\Omega^{(l)},N\right)}\le|||\mathcal{U}u|||_{\mathcal{H}\left(\Omega,\Gamma^{(p)},\Gamma\right)}\le C\,|||u|||_{\mathcal{H}\left(\Omega^{(l)},N\right)}\,, \tag{15}$$

where C is a positive number independent of u and λ.

The proof of this theorem follow from some lemmas corresponding to model problems in $\mathbf{R}^n$ and $\mathbf{R}^n_+$. Let us consider the following operators

$$L_{l0}\left(D,\lambda\right)=\left\{\sum_{|\alpha|=2m}a^{ij}_{l\alpha}D^\alpha-\lambda\omega^{ij}_l\delta_{ij}\right\}_{i,j=1}^{N}\quad\left(l=1,\dots,q+1\right);$$

(δ_{ij} denotes the symbol of Kronecker) and $B_{k0}\left(D\right)=\left\{\sum_{|\beta|=m_k}b^{kj}_\beta D^\beta\right\}$ ($k=1,\dots,Nm$) with constant coefficients $a^{ij}_{l\alpha}$, ω^{ij}_l and b^{kj}_β respectively. For convenience, we assume that $\partial\Omega_{q+1}\cap\Gamma\neq\emptyset$.

The following two results are from (Agranovish and Vishik 1964, Roitberg and Serdyuk 1991).

Lemma 1 *Suppose that the operators $L_{l0}(D,\lambda)$ satisfy Assumption 2(i). Then for $|\lambda|$ sufficiently large and any $f \in L_2(\mathbf{R}^n, N)$, the system of equations*

$$L_{l0}(D,\lambda)\,u(x) = f(x) \quad \text{in } \mathbf{R}^n \ \ (l = 1,\dots,q+1) \tag{16}$$

has one and only one solution $u \in H_{2m}(\mathbf{R}^n, N)$ and there exists a constant $C > 0$ not depending on u and λ such that the following estimate holds

$$C^{-1}\,|||u|||_{2m,\mathbf{R}^n} \le ||L_{l0}u||_{0,\mathbf{R}^n} \le C\,|||u|||_{2m,\mathbf{R}^n} \ \ (l = 1,\dots,q+1)\,. \tag{17}$$

Lemma 2 *Suppose that $L_{(q+1)0}$ satisfies Assumption 2(i) and together with the B_{k0}'s Assumption 2(ii). Then for $|\lambda|$ sufficiently large, for each $f \in L_2\left(\mathbf{R}^n_+, N\right)$ and $g_k \in H_{2m-m_k-\frac{1}{2}}\left(\mathbf{R}^{n-1}\right)$ ($k = 1,\dots,Nm$), the problem in the half-space*

$$L_{(q+1)0}(D,\lambda)\,u(x) = f(x) \quad \text{in } \mathbf{R}^n_+, \tag{18}$$

$$B_{k0}(D)\,u(x)\,|_{x_n=0} = g_k(x') \quad (k = 1,\dots,Nm) \tag{19}$$

is uniquely solvable in $H_{2m}\left(\mathbf{R}^n_+, N\right)$ and there exists a positive constant C independent of u and λ, such that

$$\begin{aligned} C^{-1}\,|||u|||_{2m,\mathbf{R}^n_+} &\le \left|\left|L_{(q+1)0}u\right|\right|_{0,\mathbf{R}^n_+} + \sum_{k=1}^{Nm} |||B_{k0}u|||_{2m-m_k-\frac{1}{2},\mathbf{R}^{n-1}} \\ &\le C\,|||u|||_{2m,\mathbf{R}^n_+}\,. \end{aligned} \tag{20}$$

We prove

Lemma 3 *Suppose that the operators L_{l0} and $L_{l'0}$ are given in $\mathbf{R}^n_+$ and $\mathbf{R}^n_-$ respectively, satisfy Assumption 2(i) and together with D^μ ($\mu = 0,\dots,2m-1$) Assumption 2(iii). Then for $|\lambda|$ sufficiently large, for each $f = (f_l, f_{l'}) \in \mathcal{L}_2(\mathbf{R}^n, N) = L_2\left(\mathbf{R}^n_+, N\right) \times L_2\left(\mathbf{R}^n_-, N\right)$ and $h_\mu \in H_{2m-\mu-\frac{1}{2}}\left(\mathbf{R}^{n-1}, N\right)$, the problem*

$$L_{l0}(D,\lambda)\,u_l(x) = f_l(x) \quad \text{in } \mathbf{R}^n_+\,, \tag{21}$$

$$L_{l'0}(D,\lambda)\,u_{l'}(x) = f_{l'}(x) \quad \text{in } \mathbf{R}^n_-, \tag{22}$$

$$[D^\mu_n u(x',0)] = D^\mu_n u_l(x',0) - D^\mu_n u_{l'}(x',0) = h_\mu(x') \ \ (\mu = 0,\dots,2m-1)\,, \tag{23}$$

is uniquely solvable in $\mathcal{H}_{2m}(\mathbf{R}^n, N) = H_{2m}\left(\mathbf{R}^n_+, N\right) \times H_{2m}\left(\mathbf{R}^n_-, N\right)$ and there exists a constant C not depending on u and λ, such that for a solution $u \in \mathcal{H}_{2m}(\mathbf{R}^n, N)$ of problem (21)-(23) the following apriori estimate holds

$$\begin{aligned} C^{-1}\,|||u|||_{\mathcal{H}_{2m}(\mathbf{R}^n,N)} &\le |||f|||_{\mathcal{L}_2(\mathbf{R}^n,N)} + \sum_{\mu=0}^{2m-1} |||h_\mu|||_{2m-\mu-\frac{1}{2},\mathbf{R}^{n-1}} \\ &\le C\,|||u|||_{\mathcal{H}_{2m}(\mathbf{R}^n,N)}\,. \end{aligned} \tag{24}$$

Proof. We derive the assertions of the lemma from Lemma 2 by reducing the problem (21)-(23) to a problem in the half-space $\mathbf{R}_+^n$. In fact setting

$$\tilde{u} = \left(u_l, \tilde{u}_{l'}\right); \; \tilde{u}_{l'}\left(x', x_n\right) = u_{l'}\left(x', -x_n\right) \tag{25}$$

in (21)-(23), we obtain the problem

$$\tilde{L}_0\left(D, \lambda\right) \tilde{u}\left(x\right) = \tilde{f}\left(x\right) \quad \text{in } \mathbf{R}_+^n, \tag{26}$$

$$B_\mu\left(D\right) \tilde{u}\left(x', 0\right) = h_\mu\left(x'\right) \;\; \left(\mu = 0, \dots, 2m-1\right), \tag{27}$$

where

$$\tilde{L}_0\left(D, q\right) = \begin{pmatrix} L_{l0}\left(D, \lambda\right) & 0 \\ 0 & \tilde{L}_{l'0}\left(D, \lambda\right) \end{pmatrix}$$

is a square matrix of dimension $2N$;

$$\tilde{L}_{l0}\left(D, q\right) = \left\{ \sum_{|\alpha|=2m} \left(-1\right)^{\alpha_n} a_{l'\alpha}^{ij} D^\alpha - \lambda \omega_{l'}^{ij} \delta_{ij} \right\}_{i,j=1}^N,$$

$$B_\mu\left(D\right) = \left(D_n^\mu E, \left(-1\right)^{\mu+1} D_n^\mu E\right) \;\left(\mu = 0, \dots, 2m-1\right);$$

E is the unit $N \times N$ matrix,

$$\tilde{f}\left(x\right) = \left(f_l\left(x\right), \tilde{f}_{l'}\left(x\right)\right); \; \tilde{f}_{l'}\left(x\right) = f_{l'}\left(x', -x_n\right).$$

Since the problem (21)-(23) is elliptic with a parameter, it follows that (26)-(27) is also elliptic with a parameter. Thus from Lemma 2 we have that for $|\lambda|$ sufficiently large the problem (26)-(27) is uniquely solvable in $H_{2m}\left(\mathbf{R}_+^n, 2N\right)$ and the solutions $\tilde{u}$ belonging to this space satisfy the apriori estimate

$$\begin{aligned} C_1^{-1} \left\{ \left|\left|\tilde{f}\right|\right|_{0,\mathbf{R}_+^n} + \sum_{\mu=0}^{2m-1} |||h_\mu|||_{2m-\mu-\frac{1}{2},\mathbf{R}_+^n} \right\} &\leq |||\tilde{u}|||_{2m,\mathbf{R}_+^n} \\ &\leq C_1 \left\{ \left|\left|\tilde{f}\right|\right|_{0,\mathbf{R}_+^n} + \sum_{\mu=0}^{2m-1} |||h_\mu|||_{2m-\mu-\frac{1}{2},\mathbf{R}^{n-1}} \right\}, \end{aligned} \tag{28}$$

the constant C_1 is independent of $\tilde{u}$ and λ. Let rewrite the second inequality in (28) in the form

$$\begin{aligned} & |||u_l|||_{2m,\mathbf{R}_+^n} + |||\tilde{u}_{l'}|||_{2m,\mathbf{R}_+^n} \\ \leq \; & C_1 \left\{ ||f_l||_{0,\mathbf{R}_+^n} + \left|\left|\tilde{f}_{l'}\right|\right|_{0,\mathbf{R}_+^n} + \sum_{\mu=0}^{2m-1} |||h_\mu|||_{2m-\mu-\frac{1}{2},\mathbf{R}^{n-1}} \right\}. \end{aligned} \tag{29}$$

Making the change of variable $x_n \to -x_n$ in the norms of the functions $\tilde{u}_{l'}$ and $\tilde{f}_{l'}$ in (29), we get

$$|||u|||_{\mathcal{H}_{2m}(\mathbf{R}^n,N)} \leq C_1 \left\{ ||f||_{\mathcal{L}_2(\mathbf{R}^n,N)} + \sum_{\mu=0}^{2m-1} |||h_\mu|||_{2m-\mu-\frac{1}{2},\mathbf{R}^{n-1}} \right\}; \tag{30}$$

This is the first inequality in (24). From it, follows the uniqueness of a solution (provided it exists) of (21)-(23). The proof of the second inequality in (24) follows similarly. It is clear that the existence of a solution to problem (21)-(23) follows from the existence of a solution to (26)-(27) as seen from the following commutative diagram

$$\begin{array}{ccc} \mathcal{H}_{2m}\left(\mathbf{R}^n,N\right) & \overset{\mathbf{U}}{\rightarrow} & \mathcal{H}\left(\mathbf{R}^n\right) = \mathcal{L}_2\left(\mathbf{R}^n,N\right) \times \prod_{\mu=0}^{2m-1} H_{2m-\mu-\frac{1}{2}}\left(\mathbf{R}^{n-1},N\right) \\ \downarrow\ r_1 & & \downarrow\ r_2 \\ H_{2m}\left(\mathbf{R}^n_+,2N\right) & \overset{\mathbf{U}'}{\rightarrow} & H\left(\mathbf{R}^n_+\right) = L_2\left(\mathbf{R}^n_+,2N\right) \times \prod_{\mu=0}^{2m-1} H_{2m-\mu-\frac{1}{2}}\left(\mathbf{R}^{n-1},N\right) \end{array}, \tag{31}$$

where r_1 is the reflexion about the hyperplane $\mathbf{R}^{n-1}$ associating $u \in \mathcal{H}_{2m}\left(\mathbf{R}^n,N\right)$ with the function $\tilde{u} \in H_{2m}\left(\mathbf{R}^n_+,2N\right)$ defined by (25), r_2 is also a reflexion about the hyperplane $\mathbf{R}^{n-1}$ mapping $\mathcal{H}\left(\mathbf{R}^n\right)$ onto $H\left(\mathbf{R}^n_+\right)$, $\mathbf{U}'$ is the operator assigning to a solution $\tilde{u} \in H_{2m}\left(\mathbf{R}^n_+,2N\right)$ of problem (26)-(27) the row $\left(\tilde{f},h_0,\ldots,h_{2m-1}\right) \in H\left(\mathbf{R}^n_+\right)$; from the unique solvability of (26)-(27) it is clear that $\mathbf{U}'$ is one-to-one and bounded (from (28)), $\mathbf{U}$ is the operator assigning to a solution $u \in \mathcal{H}_{2m}\left(\mathbf{R}^n,N\right)$ of problem (21)-(23) the row $(f,h_0,\ldots,h_{2m-1}) \in \mathcal{H}\left(\mathbf{R}^n\right)$. We set $\mathbf{U} = r_2^{-1} \circ \mathbf{U}' \circ r_1$; $\mathbf{U}$ is obviously a bounded (from (24)) and invertible operator. This completes the proof of the lemma. □

Now the proof of Theorem 1 follows from Lemmas 1, 2 and 3 by using a sufficiently fine partition of the unity and arguing exactely as in (Agranovich-Vishik 1964, Chap. 4 and 5), with the obvious changes. This process is lengthy but technically simple.

Remark 1 From Banach's open mapping theorem, it follows from Theorem 1 that the operator $\mathcal{U}$ defined in (12) establishes an isomorphism between the spaces (13)-(14). Further, following (Agranovich-Vishik 1964, Chap. 4) it can be easily shown that for $|\lambda|$ sufficiently large, the validity of the apriori estimate (15) for all $u \in \mathcal{H}_{2m}(\Omega,N)$ implies the fullfilment of Assumptions 2 (i), (ii), (iii).

4 Main results

Let $\mathcal{A}$ be the unbounded operator with the domain

$$\mathcal{D}\left(\mathcal{A}\right) = \left\{u \in H_{2m}\left(\Omega,N\right) : \ B_k u = 0 \ (k = 1,\ldots,Nm)\right\}$$

acting by $\mathcal{A}u = A(x,D)\,u$, for all $u \in \mathcal{D}(\mathcal{A})$, and T the bounded operator induced by the multiplication by the matrix E_ω in$L_2(\Omega,N)$. Clearly the domain of the operator $\mathcal{A}$ is dense in $L_2(\Omega,N)$. We rewrite problem (1) in the form

$$T^{-1}\mathcal{A}u = \lambda u,\ u \in \mathcal{D}(\mathcal{A}) \tag{32}$$

and call λ an eigenvalue of problem (1) if λ is an eigenvalue of (32) and refer to the root vectors of $T^{-1}\mathcal{A}$ corresponding to λ as those of (1) corresponding to λ. Lastly let $S(\lambda) = \mathcal{A} - \lambda T$ be the pencil acting in $L_2(\Omega,N)$ with the domain $\mathcal{D}(S) = \mathcal{D}(\mathcal{A})$ and $\rho(S)$ the set of its regular points. Under the conditions of Theorem 1, it is clear that the operators $\mathcal{A}$ and $T^{-1}\mathcal{A}$ are closed.
As a consequence of Theorem 1, we have the following resolvent estimate, of crucial importance for our investigations.

Theorem 2 *Let the Assumptions 1 and 2 be satisfied. Then there exists a positive number C such that $\lambda \in \rho(S)$ and*

$$\left|\left|S^{-1}(\lambda)\right|\right|_{L_2(\Omega,N)\to L_2(\Omega,N)} \le C\,|\lambda|^{-1} \tag{33}$$

for $\lambda \in \Xi(\theta_1,\theta_2)$ and sufficiently large in modulus.

Proof. Let the boundary conditions (10) and (11) be homogeneous. Thus from Theorem 1, it follows that the operator $(A(x,D) - \lambda E_\omega(x))$ establishes an isomorphism between $H_{2m}(\Omega,N)$ and $L_2(\Omega,N)$ and the estimate

$$||u||_{2m,\Omega} + |\lambda|\,||u||_{0,\Omega} \le C\,||(A(x,D) - \lambda E_\omega(x))\,u||_{0,\Omega} \tag{34}$$

holds for $u \in H_{2m}(\Omega,N)$, with $|\lambda|$ sufficiently large; C is a positive constant independent of u and λ. Furthermore the operator $\mathcal{A}$ is closed. Now the estimate (33) immediately follows from the inequality (34) and the closed graph theorem. □

Assume that $0 \in \rho(S)$ (this is always possible; if necessary by a shift in the spectral parameter λ) . We are now in the position to establish our main results. We have

Theorem 3 *Let Assumptions 1 be satisfied Suppose that Assumptions 2 are also satisfied along certain rays $\Xi(\theta_j)$ $(j = 1,\dots,k)$ in the complex plane emanating from the origin and making an angle θ_j with the positive real axis, and in this connection let the maximal angle between successive rays not exceed $2m\pi/n$.*

Then the spectrum of problem (1) is discrete and its root vectors are complete in $L_2(\Omega,N)$.

Proof. Under the conditions of the theorem, it is clear that the $\Xi(\theta_j)$'s are rays of minimal growth of the operator $\left(T^{-1}\mathcal{A} - \lambda E\right)^{-1}$, i.e., this operator exists for $\lambda \in \Xi(\theta_j)$, satisfies the inequality

$$\left|\left|\left(T^{-1}\mathcal{A} - \lambda E\right)^{-1}\right|\right| \le \text{const}\ |\lambda|^{-1} \tag{35}$$

(from Theorem 2) for $|\lambda|$ sufficiently large and is compact in $L_2(\Omega, N)$. This implies the first assertion of the theorem. Furthermore $\left(T^{-1}\mathcal{A}-\lambda E\right)^{-1}$ belongs to the Von Neuman-Schatten class $\mathbf{C}_{\frac{n}{2m}+\varepsilon}$; $\varepsilon>0$. Thus, the completeness of the root vectors of problem (1) follows from the inequality (35) and (Dunford and Schwartz 1963, Chapter XI, Sect. 9, Corollary 31). $\square$

In order to formulate our next result we state an abstract result on the summability by Abel's method introduced by (Lidskij 1962).

Summability in the sense of Abel-Lidskij. Let X be a separable Hilbert space and A an unbounded linear operator in X with a dense domain of definition $\mathcal{D}(A)$ in X having a discrete spectrum $\sigma(A)$. To each vector $f\in X$ we associate its formal Fourier series in the root vectors e_{sj} corresponding to the eigenvalues λ_s (with multiplicity m_s) of the operator A,

$$\sum_{s=1}^{\infty}\sum_{j=1}^{m_s} c_{sj}e_{sj}, \tag{36}$$

where the Fourier coefficients $c_{sj}=(f,g_{sj})$ $\{g_{sj}\}$ ($j=1,2,\ldots,m_s$) is a system biorthogonal to $\{e_{sj}\}$ ($j=1,\ 2,\ldots,\ m_s$). In general even if the root vectors of A are complete in X, the series (36) may be divergent. Let us impose on the operator A the following

Condition (α, θ). Let R be a positive number. We assume that all the eigenvalues λ_s of the operator A except a finite number lie in the disjoint angular sectors

$$\Xi_{p,R}=\{\lambda:\ |\arg\lambda-\theta_p|<\alpha_p,\ |\lambda|>R\}\ (\ p=1,2,\ldots,P\ ). \tag{37}$$

Definition 2 *We shall say that the system of root vectors $\{e_s\}$ of the operator A is summable by the method $\mathcal{A}(X,\theta_p,\beta_p)$ in the sequence $\{\lambda_s\}$ if for any vector $f\in X$, there exists a monotonely increasing sequence R_l ($l=1,2,\ldots$); $R_1=0$ tending to ∞ such that for all $t>0$, the series*

$$\sum_{p=1}^{P}\sum_{l=1}^{\infty}\sum_{\substack{s=1\\ \lambda_s\in\Xi_{p,R}\\ R_l\le|\lambda_s|\le R_{l+1}}}^{\infty}\sum_{j=1}^{m_s} c_{sj}^{(p)}e_{sj}, \tag{38}$$

converges, where the coefficients in (38) are determined by evaluating the integral

$$-\frac{1}{2\pi i}\int_{\gamma_s} e^{-\lambda^{\beta_p}t}\,(A-\lambda I)^{-1}\,fd\lambda,$$

where $\lambda^{\beta_p}=|\lambda|^{\beta_p}e^{i\beta_p\arg\lambda}$ (with $\beta_p\alpha_p\le\frac{\pi}{2}$) and γ_s is a contour about a single corresponding eigenvalue λ_s.

For $\lambda \in \rho(A)$, let $R(\lambda, A) = (A - \lambda I)^{-1}$ (I is the unit operator in X) be the resolvent of the operator A. Now we are in the position to formulate the theorem on the summability by the method $\mathcal{A}(X, \theta_p, \beta_p)$.

Theorem 4 *Let A be an unbounded linear operator with a dense domain of definition $\mathcal{D}(A)$ in X and having a discrete spectrum. Let s_j, $(j = 1, 2, \ldots)$ be the eigenvalues of the positive selfadjoint operator $\sqrt{RR^*}$ enumarated so that $s_1 \geq s_2 \geq \cdots \geq s_p \geq \cdots$. Let the following conditions be satisfied*

(i)

$$s_j \leq C j^{-\rho} \text{ for some } \rho > 0 \text{ and } C > 0$$

(ii) The spectrum of the operator A lies in the angular regions $\Xi_{p,R}$ ($p = 1, .2, \ldots, P$), $0 < \alpha_p < \frac{\pi\rho}{2}$ and each ray not in $\Xi_{p,R}$ is a ray of minimal growth of $R(\lambda, A)$, i.e., the following estimate holds

$$||R(\lambda, A)|| \leq C |\lambda|^{-1}, \ \lambda \notin \Xi_{p,R}.$$

Then for $\beta_p \in \left(\rho^{-1}, \frac{\pi\alpha_p^{-1}}{2}\right)$, the system of root vectors of the operator A is summable by the method $\mathcal{A}(X, \theta_p, \beta_p)$.

This theorem is a reformulation of a variant of Lidskji's theorem due to (M.S Agranovich 1977 page 345) in the case when the eigenvalues lie in more than one angle. The present formulation was inspired from (Kostyuchenko and Radzievskij 1974, Theorem 1) and (Kozhevnikov and Yakubov 1995 p. 229-230). We note that when $p = 1$ and $\theta_p = 0$, the definition 2 and the theorem 4 coincides with those introduced in Lidskii's original paper.

Now we have

Theorem 5 *Under the assumptions of Theorem 3, the system of root vectors of problem (1) is summable by the method $\mathcal{A}(L_2(\Omega, N), \alpha_j, \beta_j)$ ($\alpha_j = |\theta_{j+1} - \theta_j|$, $j = 1, \ldots, k-1$) for $\beta_j \in \left(\frac{n}{2m}, \frac{\pi}{\alpha_j}\right)$.*

Proof. Since $\left(T^{-1}\mathcal{A} - \lambda E\right)^{-1}$ is of class $\mathbf{C}_{\frac{n}{2m}+\varepsilon}$; $\varepsilon > 0$ and Inequality (35) holds on the rays $\Xi(\theta_j)$ ($j = 1, \ldots, k$), the affirmation the theorem is an immediate consequence of Theorem 4. □

The following result deals with the angular distribution of the eigenvalues of problem (1).

Theorem 6 *Let Assumptions 1 be satisfied and suppose that there exist the rays $\{\Xi(\theta_j)\}$ ($j = 1, 2$) in the complex plane such that*

(i) $0 < \theta_2 - \theta_1 < \min\{2\pi, 2m\pi/n\}$;

(ii) Assumptions 2 are satisfied for $\theta = \theta_j$ ($j = 1, 2$).

Suppose also that for some θ' ($\theta_1 < \theta' < \theta_2$) at least one of the assumptions 2 (i), (ii) or (iii) is violated for $\theta = \theta'$.

Then there are infinitely many eigenvalues of the problem (1) in the sector $\theta_1 < \arg\lambda < \theta_2$.

This theorem is proved exactely as in (Agmon 1962 Theorem 3.3).

Example. Suppose that in (1), the operator $A(x, D)$ is strongly elliptic and the boundary conditions are of Dirichlet type. Then the assumptions 2 (i), (ii) are immediately satisfied (the set of the corresponding λ's depends on the sign of the diagonal entries of E_ω in the region adjacent to Γ). The proof is similar to (Agmon, Douglis and Nirenberg 1964, Agranovish and Vishik 1964 (Chap. 4)) where the question of ellipticity for the Dirichlet problem for strongly elliptic systems has been considered. Following the same arguments as in these papers, one can show that if at least one of the diagonal entries of the weight matrix E_ω changes sign inside Ω, then assumption 2 (iii) is satisfied only when $\arg\lambda = \pm\frac{\pi}{2}$. Thus the revolvent set of the pencil $S(\lambda)$ is located on the imaginary axis in the complex plane and the rays $\arg\lambda = \pm\frac{\pi}{2}$ are the rays of minimal growth of $S^{-1}(\lambda)$. Furthermore from theorem 3 the spectrum of $S(\lambda)$ is discrete and is located in the half-planes $\mathrm{Im}\lambda < 0$ and $\mathrm{Im}\lambda > 0$. We obtain also that when $2m > n$, the root vectors are complete in $L_2(\Omega, N)$ and summable by the method $\mathcal{A}(L_2(\Omega, N), \alpha_j, \beta_j)$ for $\beta_j \in \left(\frac{n}{2m}, 1\right)$ $(j = 1, 2)$; here $\alpha_j = \pi$.

Remark 2 When the restriction on the order of the boundary operators $\{B_k\}_1^{Nm}$, $m_k < 2m$ is lifted, the investigation of the completeness of root vectors becomes more complicated. But following (Agranovich 1990), we can obtain the completeness of root vectors in spaces defined by boundary conditions. Let s be an integer $\geq \max\{2m,\ m_1 + 1, \ldots, m_{Nm} + 1\}$. We consider the unbounded operator $\mathcal{A}$ with the domain

$$\mathcal{D}(\mathcal{A}) = \{u \in H_s(\Omega, N) : B_k u = 0\ (k = 1, \ldots, Nm)\}, \tag{39}$$

acting by $\mathcal{A}u = A(x, D)u$ for all $u \in \mathcal{D}(\mathcal{A})$. We have $\mathcal{A}u \in H_{s-2m}(\Omega, N)$ but $\mathcal{A}$ may fail to be densely defined in this space. In this situation, under the fulfilment of Assumptions 1 and 2 and the corresponding smoothness conditions on the coefficients and the domains ((i)$a_\alpha^{ij}(x) \in \mathbf{C}^{s-2m}\left(\bar{\Omega}\right)$ for $|\alpha| = 2m$ and $D^\mu a_\alpha^{ij}(x) \in L_\infty(\Omega)$ for $|\alpha| < 2m$ ($|\mu| \leq s - 2m$), (ii) $b_\beta^{kj}(x) \in C^{s-m_k}(\Gamma)$ for $|\beta| = m_k$ and $D^\mu b_\beta^{kj}(x) \in L_\infty(\Gamma)$ for $|\beta| < m_k$ ($|\mu| \leq s - m_k$), (iii) Γ and Γ_p ($p = 1, \ldots, q$) are of class $\mathbf{C}^s$, in the conditions of conjugation (2) $j = 0, \ldots, s-1$), the resolvent $S^{-1}(\lambda)$ exists along some rays, is compact in $H_{s-2m}(\Omega, N)$ and for $|\lambda|$ sufficiently large, satisfies the estimate

$$\left|\left|S^{-1}(\lambda)\right|\right|_{H_{s-2m}(\Omega,N)\to H_{s-2m}(\Omega,N)} \leq \text{const } |\lambda|^\sigma,$$

where σ is an integer ≥ -1. Furthermore under the conditions of Theorem 3, we obtain the completeness of root vectors of problem (1) in the closure of the subspace

$$\begin{aligned}\mathcal{D}\left(\mathcal{A}^{\sigma+2}\right) &= \Big\{u \in H_s\left(\Omega, N\right) : B_k A^{\sigma+1} u = 0 \text{ on } \Gamma \ (k = 1, \ldots, Nm), \\ &\qquad s - m_k - 2m(\sigma+1) > \frac{1}{2},\ s - 2m(\sigma+2) \geq 0\Big\}; \\ \sigma &= -1, 0, 1, \ldots\end{aligned}$$

in $H_{s-2m}\left(\Omega, N\right)$, and in $L_2\left(\Omega, N\right)$. The proof follows as in (Agranovish 1990), where more general problems have been considered.

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