Electronic Journal of Differential Equations, Vol. **1998**(1998) No. 18, pp. 1–6.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp (login: ftp) 147.26.103.110 or 129.120.3.113

SOME REMARKS ON A SECOND ORDER EVOLUTION EQUATION

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Abstract

We prove the strong asymptotic stability of solutions to a second order evolution equation when the LaSalle's invariance principle cannot be applied due to the lack of monotonicity and compactness.

§1. Introduction and statement of the main result

In recent papers [1, 2] we studied the asymptotic stability for some dissipative wave systems. Earlier work in the same direction is due to Nakao [7] who treated particularly the case of abstract evolution equations. In this work we give a new asymptotic stability theorem which extends the analysis in [5, 8] by taking into account the new approach introduced in [1, 2].

We focus on abstract equations of the form

$$\begin{gathered} u'' - \operatorname{div}\big((1+|\nabla u|^a)^b |\nabla u|^{c-2}\nabla u\big) + g(u') = 0 \quad \text{in} \quad \Omega \times \mathbb{R}_+, \\ u(x,0) = u_0(x), u'(x,0) = u_1(x) \quad \text{in} \quad \Omega, \\ u(x,t) = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}_+, \end{gathered} \tag{P}$$

where Ω is a domain in $\mathbb{R}^n$ of *finite measure* with smooth boundary $\partial\Omega$ and $a \geq 1$, $b, c > 1$ are real numbers such that $ab + c \geq 1$. Concrete examples of (P) include the dissipative wave equation

$$\begin{gathered} u'' - \Delta u + g(u') = 0 \quad \text{in} \quad \Omega \times \mathbb{R}_+, \\ u(x,0) = u_0(x), u'(x,0) = u_1(x) \quad \text{in} \quad \Omega, \\ u(x,t) = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}_+, \end{gathered} \tag{$P1$}$$

when $a = b = 0$, $c = 2$. The degenerate Laplace operator

$$\begin{gathered} u'' - \operatorname{div}\big(|\nabla u|^{c-2}\nabla u\big) + g(u') = 0 \quad \text{in} \quad \Omega \times \mathbb{R}_+, \\ u(x,0) = u_0(x), u'(x,0) = u_1(x) \quad \text{in} \quad \Omega, \\ u(x,t) = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}_+, \end{gathered} \tag{$P2$}$$

1991 *Subject Classification:* 35B37, 35L70, 35B40.
Key words and phrases: asymptotic behavior, strong stabilization.

Submitted April 15, 1998. Published July 2, 1998.

when $a=b=0$, $c>1$. And the quasilinear wave equation

$$\begin{gathered} u'' - \operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) + g(u') = 0 \quad \text{in} \quad \Omega\times\mathbb{R}_+, \\ u(x,0)=u_0(x), u'(x,0)=u_1(x) \quad \text{in} \quad \Omega, \\ u(x,t)=0 \quad \text{on} \quad \partial\Omega\times\mathbb{R}_+, \end{gathered} \tag{P3}$$

when $a=2$, $b=-1/2$ and $c=2$. Problem (P3), with $-\Delta u'$ instead of $g(u')$, describes the motion of fixed membrane with strong viscosity. This problem with $n=1$ was proposed by Greenberg [3] and Greenberg-MacCamy-Mizel [4] as a model of quasilinear wave equation which admits a global solution for large data. Quite recently, Kobayashi-Pecher-Shibata [6] have treated such nonlinearity and proved the global existence of smooth solutions. Subsequently, Nakao [8] has derived a decay estimate of the solutions under the assumption that the mean curvature of $\partial\Omega$ is non-positive. The object of this paper is to study the asymptotic behavior of the solution u of (P) which is assumed to exist in the class

$$u \in C(\mathbb{R}_+, W_0^{1,ab+c}(\Omega)) \cap C^1(\mathbb{R}_+, L^2(\Omega)) \tag{1.1}$$

without any boundedness or geometrical conditions on Ω.

We make the following assumptions on the nonlinear function g:

(H1) $g:\mathbb{R}\to\mathbb{R}$ is locally Lipschitz continuous

(H2) $xg(x)>0$ for all $x\neq 0$

(H3) There exists a number $q\geq 1$ satisfying

$$\begin{gathered} (n-2)q \leq n+2 \quad \text{for} \quad (P1) \\ (n-c)q \leq n(c-1)+c \quad \text{for} \quad (P2) \\ (n-1)q \leq 1 \quad \text{for} \quad (P3)\,, \end{gathered}$$

and there exist positive constants c_1, c_2 such that

$$c_1|x| \leq |g(x)| \leq c_2|x|^q \quad \text{for all} \quad |x|\geq 1\,.$$

We define the energy associated to the solution given by (1.1) by the following formula

$$E(u(t)) := \frac{1}{2}\|u'(t)\|_2^2 + \|\mathcal{A}(\nabla u)\|_1\,, \tag{1.2}$$

where $\frac{\partial\mathcal{A}(v)}{\partial v} := (1+|v|^a)^b|v|^{c-2}v$.

Our main result is the following

Main Theorem. *It holds that*

$$E(u(t)) \to 0\,, \quad \text{as} \quad t\to+\infty\,,$$

for every solution u satisfying (1.1).

§2. Proof of the main theorem

For the proof we need the two following lemmas.

Lemma 2.1. *It holds that*

$$\int_0^t \int_\Omega |ug(u')|\,dx\,ds = o(t)\,, \quad t \to +\infty\,.$$

Lemma 2.2. *It holds that*

$$\int_0^t \int_\Omega |u'|^2\,dx\,ds = o(t)\,, \quad t \to +\infty\,.$$

Proof of lemma 2.1. *As g is locally Lipschitz continuous we have*

$$\begin{aligned}\int_{|u'|\le 1} |ug(u')|\,dx \le& c\int_\Omega (|u'|\,|g(u')|)^{1/2}|u|\,dx\\ \le& c\Big(\int_\Omega u'g(u')\,dx\Big)^{1/2}\|u\|_{L^2(\Omega)}\,.\end{aligned}$$

Similarly, by (H3) we have

$$\int_{|u'|>1} |ug(u')|\,dx \le c\Big(\int_\Omega u'g(u')\,dx\Big)^{\frac{1}{(q+1)'}}\|u\|_{L^{q+1}(\Omega)}$$

where $(q+1)' = \frac{q}{q+1}$ is the Hölder conjugate of $q+1$.

Then from the Hölder's inequality we obtain

$$\begin{aligned}&\int_0^t \int_\Omega |ug(u')|\,dx\,ds\\ &\qquad\le c\Big(\int_0^t \int_\Omega u'g(u')\,dx\,ds\Big)^{1/2}\sqrt{t}\sup_{[0,t]}\|u(s)\|_{L^2(\Omega)}\\ &\qquad\quad + ct^{\frac{1}{q+1}}\Big(\int_0^t \int_\Omega u'g(u')\,dx, ds\Big)^{\frac{1}{(q+1)'}}\sup_{[0,t]}\|u(s)\|_{L^{q+1}(\Omega)}\,.\end{aligned}$$

Using the Hölder, Sobolev, and Poincaré inequalities we have

$$\|u(s)\|_{L^2(\Omega)} \le c\,\|u(s)\|_{L^{q+1}(\Omega)} \le c\,E(s)^{1/2} \le c\,E(0)^{1/2} \quad \text{for all} \quad s \ge 0.$$

From these estimates it follows that

$$\int_0^t \int_\Omega |ug(u')|\,dx, ds \le c\sqrt{t} + ct^{\frac{1}{q+1}} = o(t), \quad t \to +\infty\,.$$

Proof of lemma 2.2. *Let $\varepsilon > 0$ be an arbitrarily small real and set*

$$M(\varepsilon) = \sup \{\frac{x}{g(x)}; \quad |x| \geq \sqrt{\frac{\varepsilon}{|\Omega|}} \}$$

by hypotheses (H1)-(H3), we have $M(\varepsilon) < +\infty$.

Clearly,

$$\int_{|u'|<\sqrt{\frac{\varepsilon}{|\Omega|}}} |u'|^2 \, dx \leq \varepsilon.$$

On the other hand

$$\int_{|u'|\geq\sqrt{\frac{\varepsilon}{|\Omega|}}} |u'|^2 \, dx = \int_{|u'|\geq\sqrt{\frac{\varepsilon}{|\Omega|}}} \frac{u'}{g(u')} \, u'g(u') \, dx \leq M(\varepsilon) \int_\Omega u'g(u') \, dx \, .$$

As

$$\int_{|u'|\geq\sqrt{\frac{\varepsilon}{|\Omega|}}} |u'|^2 \, dx \leq \sqrt{2E(0)} \, \Big(\int_{|u'|\geq\sqrt{\frac{\varepsilon}{|\Omega|}}} |u'|^2 \, dx\Big)^{1/2},$$

we deduce that

$$\int_\Omega |u'|^2 \, dx \leq \varepsilon + \sqrt{2E(0)M(\varepsilon)} \, \Big(\int_\Omega u'g(u') \, dx\Big)^{1/2},$$

and then by the Hölder inequality

$$\begin{aligned} \int_0^t \int_\Omega |u'|^2 \, dx \, ds \leq & \varepsilon t + \sqrt{2E(0)M(\varepsilon)} \, \sqrt{t} \, \Big(\int_0^t \int_\Omega u'g(u') \, dx, ds\Big)^{1/2} \\ \leq & \varepsilon t + E(0)\sqrt{2M(\varepsilon)} \, \sqrt{t} = o(t), \quad t \to +\infty \, . \end{aligned}$$

Proof of the main theorem

Assume on the contrary that $l := \lim_{t\to+\infty} E(t) > 0$. Then we have

$$\int_0^t \int_\Omega uu' \, dx \, ds = \int_0^t \int_\Omega |u'|^2 - A(\nabla u)\nabla u - g(u')u \, dx \, ds$$

where $A(\nabla u) := (1 + |\nabla u|^a)^b |\nabla u|^{c-2} \nabla u$. Following the approach introduced in [1, 2], we shall prove that

$$\|u'\|_2^2 + \int_\Omega A(\nabla u)\nabla u \, dx \geq c_3 > 0 \, . \tag{2.1}$$

We have

$$\|u'(t)\|_2^2 + 2\|\mathcal{A}(\nabla u)\|_1 \geq l \, ;$$

hence, if $\|u'(t)\|_2^2 \geq \frac{l}{2}$ we get (2.1) with $c_3 = \frac{l}{2}$. And, if we have $\|\mathcal{A}(\nabla u)\|_1 \geq \frac{l}{4}$, then

$$c_4\Big(\|\nabla u\|_1 + \|\nabla u\|_{ab+c}^{ab+c}\Big) \geq \frac{l}{4}$$

that is

$$\|\nabla u\|_{ab+c} \geq c_5 > 0\,.$$

Since A is coercive (that is $(A(v),v)_{L^2} \geq c_6|v|^{ab+c}$ with $|v| \geq |v_0|$), we get (2.1) with a positive constant $c_7 > 0$.

Thanks to lemmas 1,2, and the relation (2.1), we arrive by the same arguments in [1, 2] to

$$\phi(t) \to -\infty \quad \text{as} \quad t \to +\infty\,,$$

where $\phi(t) = \int_\Omega uu'\,dx$. This is a contradiction to the fact that $|\phi(t)| \leq c_8 E(0)$. Thus

$$\lim_{t\to+\infty} E(t) = 0\,.$$

Remark. If g is linear or superlinear near the origin, then it is sufficient to consider a domain $\Omega \subset \mathbb{R}^n$ in which the Poincaré's inequality holds.

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