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A Multiplicity Result for a Class of Quasilinear Elliptic and Parabolic Problems *

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Abstract

We prove the existence of infinitely many solutions for a class of quasilinear elliptic and parabolic equations, subject respectively to Dirichlet and Dirichlet–periodic boundary conditions. We assume that the primitive of the nonlinearity at the right-hand side oscillates at infinity. The proof is based on the construction of upper and lower solutions, which are obtained as solutions of suitable comparison equations. This method allows the introduction of conditions on the potential for the study of parabolic problems, as well as to treat simultaneously the singular and the degenerate case.

1 Introduction and statements

Let us consider the following quasilinear elliptic and parabolic problems:

$$\begin{cases} -\operatorname{div} a(x,\nabla u) = b(x,u,\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

and

$$\begin{cases} u_t - \operatorname{div}_x a(x,\nabla_x u) = c(x,t,u,\nabla_x u) & \text{in } Q, \\ u(x,t) = 0 & \text{on } \Sigma, \\ u(x,0) = u(x,T) & \text{on } \Omega. \end{cases} \tag{1.2}$$

We assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain, with boundary $\partial\Omega$ of class C^2, and T is a fixed positive number. We set $Q := \Omega\times]0,T[$ and $\Sigma := \partial\Omega\times]0,T[$. We suppose that the coefficient function $a : \overline{\Omega}\times\mathbb{R}^N \to \mathbb{R}^N$ is such that $a(x,\xi) = \nabla_\xi A(x,\xi)$ for all $(x,\xi) \in \overline{\Omega}\times\mathbb{R}^N$, for some $A : \overline{\Omega}\times\mathbb{R}^N \to \mathbb{R}$ satisfying $A \in C^1(\overline{\Omega}\times\mathbb{R}^N,\mathbb{R})$, $A(x,\cdot) \in C^2(\mathbb{R}^N\setminus\{0\},\mathbb{R})$ for all $x \in \overline{\Omega}$ and $\nabla_\xi A(\cdot,\xi) \in C^1(\overline{\Omega},\mathbb{R})$ for all $\xi \in \mathbb{R}^N$. The following structure conditions of Leray–Lions type (cf. [12], [11], [4]) are also assumed:

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(i_1) there exist constants $p > 1, \gamma_1, \gamma_2 > 0$ and $\kappa \in [0,1]$ such that

$$\gamma_1(\kappa + |\xi|)^{p-2}\,|s|^2 \le \sum_{i,j=1}^{N} \frac{\partial^2 A}{\partial \xi_i \partial \xi_j}(x,\xi)\, s_i\, s_j \le \gamma_2(\kappa + |\xi|)^{p-2}\,|s|^2$$

for all $x \in \overline{\Omega}$, $\xi \in \mathbb{R}^N \setminus \{0\}$ and $s \in \mathbb{R}^N$,

and

(i_2) there exist constants $p > 1$ and $\gamma_3 > 0$ such that

$$\max_{i,j=1\ldots N} \left| \frac{\partial^2 A}{\partial x_i \partial \xi_j}(x,\xi) \right| \le \gamma_3(1 + |\xi|)^{p-1}$$

for all $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^N$.

It is understood that conditions (i_1) and (i_2) hold with the same exponent p. We further suppose that the functions $b : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $c : \overline{Q} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are continuous and satisfy, respectively,

(i_3) there exist two continuous functions $f, g : \mathbb{R} \to \mathbb{R}$ such that

$$f(s) \le b(x,s,\xi) \le g(s)$$

for all $x \in \overline{\Omega}$, $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$,

and

(i_4) there exist two continuous functions $f, g : \mathbb{R} \to \mathbb{R}$ such that

$$f(s) \le c(x,t,s,\xi) \le g(s)$$

for all $x \in \overline{\Omega}$, $t \in [0,T]$, $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$.

We finally set $F(s) = \int_0^s f(\sigma)d\sigma$ and $G(s) = \int_0^s g(\sigma)d\sigma$.

The aim of this paper is to establish the existence of infinitely many solutions to problems (1.1) (1.2), placing only conditions on the functions F and G, which are assumed to have an oscillatory behaviour at infinity. In this way we are able to generalize to (1.1) a similar statement recently obtained in [13] for the less general elliptic problem

$$\begin{cases} -\operatorname{div} a(\nabla u) = d(u) + e(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

as well as to extend it to the parabolic problem (1.2). In this way we obtain a result which is completely new within the framework of parabolic equations, where it is fairly unusual to introduce conditions on the potential, on account of the lack of variational structure. The study of the elliptic problem and that of the parabolic problem proceed in a quite parallel way and depend, in both cases, on the analysis of some auxiliary elliptic equations, for which some ideas introduced in [13] are employed. We stress that even the sole extension of the result in [13], from (1.3) to (1.1), is not trivial, since a step of the proof in [13] (cf. Lemma 2.2 therein) relies in an essential way on the autonomous character of the coefficient function a in (1.3).

Before stating our results, we recall that a solution of (1.1) is a function $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that

$$\int_\Omega a(x, \nabla u)\, \nabla w \, dx = \int_\Omega b(x, u, \nabla u)\, w \, dx$$

for all $w \in W_0^{1,p}(\Omega)$. Whereas, a solution of (1.2) is a function $u \in \mathcal{V}_0 \cap L^\infty(Q)$, with $\mathcal{V}_0 := L^p(0, T; W_0^{1,p}(\Omega))$, having distributional derivative $u_t \in {\mathcal{V}_0}^* + L^q(Q)$ for some $q > 1$, which satisfies $u(0) = u(T)$ and

$$< u_t, w > + \iint_Q a(x, \nabla_x u)\, \nabla_x w \, dx\, dt = \iint_Q c(x, t, u, \nabla_x u)\, w \, dx\, dt$$

for all $w \in \mathcal{V}_0 \cap L^\infty(Q)$, where $< \cdot, \cdot >$ denotes the duality pairing between ${\mathcal{V}_0}^* + L^q(Q)$ and $\mathcal{V}_0 \cap L^{q/(q-1)}(Q)$. Of course, the exponent p, which appears in these definitions, comes from (i_1) and (i_2). Note also that the convergence of the integrals at the right hand side is guaranteed by assumptions (i_3) and (i_4) on b and c, respectively, and by the boundedness of the solution u.

Theorem 1.1 *Assume (i_1), (i_2), and (i_3). Moreover, suppose that*

(j_1) $\displaystyle \liminf_{s\to+\infty} \frac{G(s)}{s^p} \leq 0,$

(j_2) $\displaystyle -\infty < \liminf_{s\to+\infty} \frac{F(s)}{s^p} \leq \limsup_{s\to+\infty} \frac{F(s)}{s^p} = +\infty,$

(j_3) $\displaystyle \liminf_{s\to-\infty} \frac{F(s)}{|s|^p} \leq 0,$

(j_4) $\displaystyle -\infty < \liminf_{s\to-\infty} \frac{G(s)}{|s|^p} \leq \limsup_{s\to-\infty} \frac{G(s)}{|s|^p} = +\infty.$

Then, problem (1.1) has two sequences $(u_n)_n$ and $(v_n)_n$ of solutions, satisfying

$$\sup_\Omega u_n \to +\infty \qquad \textit{and} \qquad \inf_\Omega v_n \to -\infty.$$

Theorem 1.2 *Assume (i_1), (i_2), (i_4), (j_1), (j_2), (j_3), and (j_4). Then problem (1.2) has two sequences $(u_n)_n$ and $(v_n)_n$ of solutions, satisfying*

$$\sup_Q u_n \to +\infty \qquad \textit{and} \qquad \inf_Q v_n \to -\infty.$$

We remark that the solutions u_n and v_n of the elliptic problem (1.1) possess more regularity since they belong to $C^{1,\sigma}(\overline{\Omega})$, for some $\sigma > 0$. Further, they satisfy the ordering condition

$$\ldots \leq v_n \leq \ldots \leq v_1 \leq u_1 \leq \ldots \leq u_n \leq \ldots \qquad \text{in } \Omega.$$

This information is only partially retained for the solutions of the parabolic problem (1.2). Indeed, we can prove that they satisfy

$$\ldots \leq v_n \leq \ldots \leq v_1 \leq u_1 \leq \ldots \leq u_n \leq \ldots \qquad \text{in } Q,$$

provided that $p \geq 2$. Anyhow, we stress that, except for this detail, we are able to obtain exactly the same multiplicity result both for the singular parabolic problem, corresponding to $1 < p < 2$, and for the degenerate one, corresponding to $p > 2$. This is due to the fact that the main tool in our proofs is the upper-and-lower-solutions method, which is a principle valid in either situation (see [6]).

At this point, it is worth commenting on the proofs of Theorems 1.1 and 1.2, which, as already pointed out, proceed in a parallel way and essentially rely on the use of the upper-and-lower- solutions method. Indeed, to get a sequence $(u_n)_n$ of solutions, with $\sup u_n \to +\infty$, the main task is to build a sequence $(\beta_n)_n$ of upper solutions and a sequence $(\alpha_n)_n$ of lower solutions, which are both of class C^1 and satisfy $\min \beta_n \geq \max \alpha_n \to +\infty$. The upper solutions β_n, both of problem (1.1) and of problem (1.2), are obtained as upper solutions of the "upper" comparison problem

$$\begin{cases} -\operatorname{div} a(x, \nabla u) = g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

These β_n are in turn constructed as solutions of the quasilinear ordinary differential equation

$$-(q\,|z'|^{p-2}\,z')' = r\,g(z) \qquad \text{in } [a, b], \tag{1.5}$$

where $]a, b[$ is the projection of Ω on, say, the x_1–axis and q, r are suitable positive weight–functions. Indeed, assumption (j_1) on G yields the existence of a sequence $(z_n)_n$ of concave, decreasing, arbitrarily-large positive solutions of (1.5). Then, the functions β_n defined by

$$\beta_n(x_1, \ldots, x_N) := z_n(x_1) \quad \text{in } \Omega$$

form a sequence of upper solutions of (1.4), satisfying $\min \beta_n \to +\infty$. Conversely, the lower solutions α_n are solutions of the "lower" comparison problem

$$\begin{cases} -\operatorname{div} a(x, \nabla u) = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

The existence of these solutions α_n is obtained using again the upper-and-lower-solutions method, applied to (1.6) and combined, as in [13], with an elementary variational argument. The upper solutions of (1.6) are the β_n previously obtained, while the lower solutions of (1.6) are constructed, using assumption (j_3), by the same argument described above applied to an equation similar to (1.5) but involving the function f. Condition (j_2) is used in order to prove that such solutions α_n of (1.6) satisfy $\max \alpha_n \to +\infty$. A completely similar argument allows us to build sequences $(v_n)_n$ of solutions of problems (1.1) and (1.2), satisfying $\inf v_n \to -\infty$.

Let us observe that the class of quasilinear differential operators we consider here includes operators of the type

$$-\mathrm{div}(|\mathcal{A}(x)\nabla u|^{p-2}\mathcal{A}^T(x)\mathcal{A}(x)\nabla u)$$

and

$$u_t - \mathrm{div}_x(|\mathcal{A}(x)\nabla_x u|^{p-2}\mathcal{A}^T(x)\mathcal{A}(x)\nabla_x u),$$

where $p > 1$ and $\mathcal{A} : \overline{\Omega} \to \mathbb{R}^{N^2}$ is a C^1 matrix–valued function, with $\mathcal{A}(x)$ nonsingular for each $x \in \overline{\Omega}$. It is clear that, if $\mathcal{A}(x)$ is the identity matrix for each x, these operators become, respectively, $-\Delta_p u$ and $u_t - \Delta_p u$, where Δ_p is the p–Laplacian with respect to the space variable. Hence, the following simple consequence of Theorem 1.2 can be stated.

Corollary 1.1 *Assume that $d : \mathbb{R} \to \mathbb{R}$ and $e : \overline{Q} \to \mathbb{R}$ are continuous functions. Moreover, suppose that*

$$\liminf_{s\to\pm\infty} \frac{D(s)}{|s|^p} = 0 \quad \textit{and} \quad \limsup_{s\to\pm\infty} \frac{D(s)}{|s|^p} = +\infty,$$

where $D = \int_0^s d(\sigma)\, d\sigma$. Then, the same conclusions of Theorem 1.2 hold for

$$\left\{\begin{array}{ll} u_t - \mathrm{div}_x(|\nabla_x u|^{p-2}\nabla_x u) = d(u) + e(x,t) & \textit{in } Q, \\ u(x,t) = 0 & \textit{on } \Sigma, \\ u(x,0) = u(x,T) & \textit{on } \Omega. \end{array}\right. \tag{1.7}$$

Actually, Theorems 1.1 and 1.2 are new, even when the differential operators are linear, as is the case of the following problems

$$\left\{\begin{array}{ll} -\mathrm{div}(\mathcal{M}(x)\nabla u) = b(x,u,\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega\,, \end{array}\right.$$

and

$$\left\{\begin{array}{ll} u_t - \mathrm{div}_x(\mathcal{M}(x)\nabla_x u) = c(x,t,u,\nabla_x u) & \text{in } Q, \\ u(x,t) = 0 & \text{on } \Sigma, \\ u(x,0) = u(x,T) & \text{on } \Omega, \end{array}\right.$$

where $\mathcal{M}(x) = \mathcal{A}^T(x)\mathcal{A}(x)$. Evidently, within this framework, one can say much more about the regularity of the solutions, which lie in $W^{2,r}(\Omega)$ for every $r > 1$, in the elliptic case, and in $W_r^{1,2}(Q)$ for every $r > 1$, in the parabolic case.

We also remark that our method can be used, in some situations, to obtain multiple solutions having a prescribed sign. We produce a model result in this direction only for the parabolic problem. It is stated for the sake of simplicity in the setting of Corollary 1.1.

Proposition 1.1 *Assume that $d : [0,+\infty[\to \mathbb{R}$ and $e : \overline{Q} \to \mathbb{R}$ are continuous functions, satisfying*

$$d(0) + e(x,t) \geq 0 \quad \textit{in } Q.$$

Moreover, suppose that

$$\liminf_{s\to+\infty} \frac{D(s)}{s^p} = 0 \quad \textit{and} \quad \limsup_{s\to+\infty} \frac{D(s)}{s^p} = +\infty,$$

where $D = \int_0^s d(\sigma)\,d\sigma$. Then, problem (1.7) has a sequence $(u_n)_n$ of solutions which satisfy

$$\inf_Q u_n = 0 \quad \textit{and} \quad \sup_Q u_n \to +\infty.$$

We further note, and it will be clear from the proofs given below, that the mere existence of a solution of problem (1.1), or respectively (1.2), without information about the multiplicity, is achieved assuming only conditions (j_1) and (j_3), in addition to (i_1), (i_2), (i_3) for problem (1.1), and (i_1), (i_2), (i_4) for problem (1.2). This statement generalizes in various directions previous existence results obtained in [8], [7], [13, Sec. 3] and [9, Th. 2.4].

We conclude by observing that more general conditions than (i_1) and (i_2) could be considered (see the beginning of the next section). Furthermore, the continuity assumptions could be replaced by suitable Carathéodory conditions.

2 Proofs

We begin by deriving some consequences of assumptions (i_1) and (i_2), which will be used in the course of this section.

First of all, it is plain that we can assume without loss of generality that

$$a(x,0) = 0 \qquad \text{in } \Omega. \tag{2.1}$$

Otherwise we could replace $a(x,\xi)$ by $\tilde{a}(x,\xi) := a(x,\xi) - a(x,0)$ and the equations in (1.1) and (1.2) by the equivalent ones

$$-\operatorname{div} \tilde{a}(x,\nabla u) = b(x,u,\nabla u) + \operatorname{div} a(x,0)$$

and, respectively,

$$u_t - \operatorname{div}_x \tilde{a}(x,\nabla_x u) = c(x,t,u,\nabla_x u) + \operatorname{div}_x a(x,0).$$

Next, observe that from (2.1) we have for all $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^N \setminus \{0\}$

$$\begin{aligned} a(x,\xi) &= a(x,\xi) - \lim_{\varepsilon\to0^+} a(x,\varepsilon\xi) \\ &= \lim_{\varepsilon\to0^+} \int_\varepsilon^1 H_\xi A(x,t\,\xi)\xi\,dt = \int_0^1 H_\xi A(x,t\,\xi)\xi\,dt, \end{aligned} \tag{2.2}$$

where $H_\xi A$ denotes the Hessian matrix of A with respect to the ξ variable. Notice that the last integral is finite due to the upper estimate in (i_1).

¿From (2.2) and the lower estimate in (i_1), we also derive for all $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^N \setminus \{0\}$, with $|\xi| \geq 1$, the inequality

$$\begin{aligned} a(x,\xi)\cdot\xi &= \int_0^1 H_\xi A(x,t,\xi)\xi\cdot\xi\,dt \\ &\geq \gamma_1 \left(\int_0^1 (\kappa + t\,|\xi|)^{p-2}\,dt\right)|\xi|^2 \geq \begin{cases} \gamma_1 2^{p-2}\,|\xi|^p & \text{if } 1<p<2 \\ \frac{\gamma_1}{p-1}|\xi|^p & \text{if } p \geq 2. \end{cases} \end{aligned}$$

Hence, we can find two constants $c_1, c_2 > 0$ such that for all $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^N$

$$a(x,\xi)\cdot\xi \geq c_1\,|\xi|^p - c_2.$$

Similarly, we can prove that for all $x \in \overline{\Omega}$ and $\xi \neq \xi' \in \mathbb{R}^N$

$$\begin{aligned} (a(x,\xi) - a(x,\xi'))\cdot(\xi-\xi') &= \int_0^1 H_\xi A(x,\xi + t\,(\xi-\xi'))(\xi-\xi')\cdot(\xi-\xi')\,dt \\ &\geq \gamma_1 \left(\int_0^1 (\kappa + |\xi + t\,(\xi-\xi')|)^{p-2}\,dt\right)|\xi-\xi'|^2 > 0. \end{aligned}$$

Using again (2.2) and the upper estimate in (i_1), we obtain for all $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^N$, with $|\xi| \geq 1$,

$$\begin{aligned} |a(x,\xi)| \leq \int_0^1 |H_\xi A(x,t\,\xi)|\,|\xi|\,dt &\leq \gamma_2 \left(\int_0^1 (\kappa + t\,|\xi|)^{p-2}\,dt\right)|\xi| \\ &\leq \begin{cases} \frac{\gamma_2}{p-1}|\xi|^{p-1} & \text{if } 1<p<2 \\ \gamma_2 2^{p-2}\,|\xi|^{p-1} & \text{if } p \geq 2. \end{cases} \end{aligned}$$

Hence, we can find two constants $c_3, c_4 > 0$ such that for all $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^N$

$$|a(x,\xi)| \leq c_3\,|\xi|^{p-1} + c_4.$$

Moreover, we can write for all $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^N$

$$A(x,\xi) = A(x,0) + a(x,0)\,\xi + \int_0^1 (1-t)\,H_\xi A(x,t\xi)\,\xi\cdot\xi\,dt,$$

where the last integral is finite by (i_1). Hence, using (i_1) and (2.1) and arguing as above, we get the existence of constants $c_5, c_6, c_7 > 0$ such that for all $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^N$

$$c_5\,|\xi|^p - c_6 \leq A(x,\xi) \leq c_7\,|\xi|^p + c_6.$$

Finally, from (i_2), we easily derive that, for each $i = 1, \ldots, N$ and for all $x, y \in \overline{\Omega}$ and $\xi \in \mathbb{R}^N$

$$\begin{aligned} |a_i(x,\xi) - a_i(y,\xi)| &\leq \sup_{(z,\xi)\in\overline{\Omega}\times\mathbb{R}^N} |\nabla_\xi\, a_i(z,\xi)|\;\delta(x,y) \\ &\leq N\,\gamma_3\,(1+|\xi|)^{p-1}\,\delta(x,y), \end{aligned}$$

where $\delta(x,y)$ denotes the geodetic distance in $\overline{\Omega}$ between x and y. Since $\partial\Omega$ is of class C^2 and therefore the geodetic distance is globally Lipschitz, we conclude that there exists a constant $c_8 > 0$ such that for all $x, y \in \overline{\Omega}$ and $\xi \in \mathbb{R}^N$

$$|a(x,\xi) - a(y,\xi)| \leq c_8\,(1+|\xi|)^{p-1}\,|x-y|.$$

Moreover, from (i_2) there exists a constant $\gamma_4 > 0$ such that for all $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^N$, with $|\xi| \geq 1$,

$$\left|\sum_{i=1}^{N} \frac{\partial a_i}{\partial x_i}(x,\xi)\right| \leq \gamma_4 |\xi|^{p-1}. \tag{2.3}$$

According to this discussion, we can conclude that conditions (0.3a), (0.3b), (0.3c), (0.3d) of Theorem 1 in [11] are satisfied, as well as conditions (A1), (A2), (A3) in [5] and (A1), (A2), (A3), (A4) in [6].

Now we state some preliminary lemmas which will eventually lead to the proof of Theorems 1.1 and 1.2. Let us consider the quasilinear elliptic problem

$$\begin{cases} -\operatorname{div} a(x,\nabla u) = h(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

where a satisfies conditions (i_1) and (i_2) and $h : \mathbb{R} \to \mathbb{R}$ is continuous. We also set $H(s) = \int_0^s h(\sigma)\,d\sigma$. Of course, the meaning of a solution of (2.4) is the same as explained in the introduction.

The first result provides the existence of an unbounded sequence of positive upper solutions of (2.4).

Lemma 2.1 *Assume that*

$$\liminf_{s\to+\infty} \frac{H(s)}{s^p} \leq 0. \tag{2.5}$$

Then, problem (2.4) *has a sequence* $(\beta_n)_n$ *of upper solutions, with* $\beta_n \in C^2(\overline{\Omega})$ *and*

$$\min_{\overline{\Omega}} \beta_{n+1} > \max_{\overline{\Omega}} \beta_n \to +\infty. \tag{2.6}$$

Proof of Lemma 1. We begin by observing that if

$$\sup\{s \geq 0 |\, h(s) \leq 0\} = +\infty,$$

then there exists a sequence $(\beta_n)_n$ of constant upper solutions with $\beta_n \to +\infty$. Therefore, we can suppose that there exists $s_0 \geq 0$ such that

$$h(s) > 0 \qquad \text{for all } s > s_0. \tag{2.7}$$

In order to build upper solutions of (2.4), we study some properties of a related one-dimensional initial value problem. Let $a < b$ be given constants and let $q, r : [a,b] \to \mathbb{R}$ be two functions, with q of class C^1 and r continuous, satisfying

$$0 < q_0 := \min_{[a,b]} q \leq \max_{[a,b]} q =: q_\infty, \tag{2.8}$$

$$\max_{[a,b]} q' \le 0 \tag{2.9}$$

and

$$0 < \min_{[a,b]} r \le \max_{[a,b]} r \;=: r_\infty . \tag{2.10}$$

Let us consider the quasilinear ordinary differential equation

$$-(q\,|u'|^{p-2}\,u')' = r\,h(u)\,. \tag{2.11}$$

By a solution of (2.11), we mean a function $u \in C^1(I)$ with $q\,|u'|^{p-2}\,u' \in C^1(I)$ on some interval $I \subset [a,b]$.

Claim 1 *For every $c > s_0$, there is $d > c$ such that* (2.11) *has a solution u, which is defined and of class C^2 on $[a,b]$ and satisfies*

$$c \le u(t) \le d, \qquad u'(t) \le -1, \qquad u''(t) \le 0 \quad \text{on } [a,b]. \tag{2.12}$$

Proof of Claim 1. Let us define the functions

$$\varphi_p(s) := \operatorname{sign}(s)|s|^{p-1}$$

and

$$\Phi_p^*(s) := \int_0^s \varphi_p^{-1}(\sigma)\,d\sigma = \int_0^s \operatorname{sign}(\sigma)\,|\sigma|^{\frac{1}{p-1}}\,d\sigma = \frac{p-1}{p}\,|s|^{\frac{p}{p-1}}.$$

Let $c > s_0$ be given and consider the initial value problem

$$\begin{cases} -(q\,\varphi_p(u'))' = r\,h(u) \\ u(a) = d \\ u'(a) = -(\frac{q_\infty}{q_0})^{\frac{1}{p-1}} \qquad \text{(i.e. } \varphi_p(u'(a)) = -\frac{q_\infty}{q_0}), \end{cases} \tag{2.13}$$

where $d > c$ is a real parameter. Since problem (2.13) is equivalent to the system

$$\begin{cases} u' = \varphi_p^{-1}(\frac{v}{q}) \\ v' = -r\,h(u) \end{cases} \tag{2.14}$$

with initial conditions

$$u(a) = d \qquad \text{and} \qquad v(a) = -q(a)\frac{q_\infty}{q_0},$$

the existence of a local solution of (2.13) and its continuability to a right maximal interval of existence are standard facts. Let us set

$$\omega := \sup\{t \in]a,b] \mid u \text{ is defined and } u > c \text{ on } [a,t]\}$$

Of course, it is $\omega > a$. Integrating (2.11) on $[a,t]$, for any $t \in]a,\omega[$, we obtain that

$$\varphi_p(u'(t)) = \frac{q(a)}{q(t)}\,\varphi_p(u'(a)) - \frac{1}{q(t)}\int_a^t r\,h(u)\,ds\,.$$

Hence, by (2.7), (2.8) and (2.10),

$$u'(t) \le -\varphi_p^{-1}\left(\frac{q(a)}{q_0}\frac{q_\infty}{q(t)}\right) \le -1. \tag{2.15}$$

¿From (2.14) and (2.15) we derive that

$$\frac{v(t)}{q_0} \le \frac{v(t)}{q(t)} \le \frac{v(t)}{q_\infty} < 0 \quad \text{on } [a,\omega[.$$

This implies that $u' = \varphi_p^{-1}(\frac{v}{q})$ is of class C^1 on $[a,\omega[$. So that $q\,\varphi_p(u') = -q\,|u'|^{p-1}$ can be differentiated. Thus, from (2.11), (2.9), (2.7) and (2.10), we obtain

$$(p-1)\,q\,|u'|^{p-2}\,u'' = q'\,|u'|^{p-1} - r\,h(u) \le 0 \quad \text{on } [a,\omega[,$$

which implies that, by (2.8),

$$u'' \le 0 \qquad \text{on } [a,\omega[\,.$$

Assume now by contradiction that

$$\omega < b \;\; (< +\infty). \tag{2.16}$$

By (2.15) there exists

$$\lim_{t\to\omega^-} u(t) = c.$$

Accordingly, we can set

$$u(\omega) := c \tag{2.17}$$

and hence u can be continued as a solution to ω.

Define, for $t \in [a,\omega]$, the energy function

$$E(t) := \frac{r_\infty}{q_\infty}\,H(u(t)) + \Phi_p^*\left(\frac{v(t)}{q_\infty}\right).$$

We easily get, for $t \in [a,\omega]$,

$$\begin{aligned} E'(t) &= \frac{r_\infty}{q_\infty}\,h(u(t))\,u'(t) + \varphi_p^{-1}\left(\frac{v(t)}{q_\infty}\right)\frac{v'(t)}{q_\infty} \\ &= \frac{r_\infty}{q_\infty}\,h(u(t))\,\varphi_p^{-1}\left(\frac{v(t)}{q(t)}\right) - \varphi_p^{-1}\left(\frac{v(t)}{q_\infty}\right)\frac{r(t)}{q_\infty}\,h(u(t)) \\ &\le \frac{r_\infty}{q_\infty}\,h(u(t))\,\varphi_p^{-1}\left(\frac{v(t)}{q_\infty}\right)\left(1 - \frac{r(t)}{r_\infty}\right) \quad \le \quad 0. \end{aligned}$$

Accordingly, since Φ_p^* is even, we have, for $t \in [a,\omega]$,

$$\begin{aligned} E(t) &= \frac{r_\infty}{q_\infty}\,H(u(t)) + \Phi_p^*\left(\frac{v(t)}{q_\infty}\right) \\ &\le \frac{r_\infty}{q_\infty}\,H(d) + \Phi_p^*\left(\frac{q(a)}{q_0}\right) \quad = \quad E(a) \end{aligned}$$

and then

$$\begin{aligned}
\left(\frac{p-1}{p}\right)\left(\frac{q_0}{q_\infty}\right)^{\frac{p}{p-1}}|u'(t)|^p &\leq \left(\frac{p-1}{p}\right)\left(\frac{q(t)}{q_\infty}\right)^{\frac{p}{p-1}}|u'(t)|^p \\
&= \Phi_p^*\left(\frac{q(t)}{q_\infty}\,\varphi_p(u'(t))\right) \\
&\leq \frac{r_\infty}{q_\infty}\,(H(d)-H(u(t)))+\Phi_p^*\left(\frac{q(a)}{q_0}\right) \\
&\leq \frac{r_\infty}{q_\infty}\,(H(d)-H(c))+\Phi_p^*\left(\frac{q_\infty}{q_0}\right).
\end{aligned}$$

Hence, by the mean value theorem, we obtain, using (2.16) and (2.17), that

$$\begin{aligned}
&|d-c|^p = |u(a)-u(\omega)|^p \;=\; |u'(\tau)|^p\,|\omega-a|^p \\
&\leq \left(\frac{p}{p-1}\right)\left(\frac{q_\infty}{q_0}\right)^{\frac{p}{p-1}}\left(\frac{r_\infty}{q_\infty}\,(H(d)-H(c))+\Phi_p^*\left(\frac{q_\infty}{q_0}\right)\right)|b-a|^p. \quad (2.18)
\end{aligned}$$

Finally, by condition (2.5), we can find a sequence $(d_n)_n$, with $d_n \to +\infty$, such that

$$\frac{H(d_n)}{{d_n}^p} \to 0.$$

Taking $d = d_n$ in (2.13) and (2.18), dividing (2.18) by ${d_n}^p$ and passing to the limit, we obtain a contradiction. This shows that $\omega = b$ and the claim follows, extending u to b as a solution.

Now, we prove the following:

Claim 2 *For each $c > s_0$, there is $d > c$ and an upper solution $\beta \in C^2(\overline{\Omega})$ of problem* (2.4) *such that*

$$c \leq \beta \leq d \qquad \text{in } \overline{\Omega}.$$

Proof of Claim 2. Let $]a,b[$ be the projection of Ω on, say, the x_1–axis and consider the quasilinear ordinary differential equation

$$-\frac{\gamma_1}{2}|u'|^{p-2}\,u''+\gamma_4\,|u'|^{p-2}\,u' = h(u) \quad \text{on } [a,b], \tag{2.19}$$

where γ_1 and γ_4 are given, respectively, in (i_1) and (2.3). By a solution of (2.19), we mean a function $u \in C^2([a,b])$, with $u' < 0$ on $[a,b]$. If we set, for $t \in [a,b]$,

$$q(t) := \exp\left(-\frac{2(p-1)\gamma_4}{\gamma_1}t\right) \quad \text{and} \quad r(t) := \frac{2(p-1)}{\gamma_1}q(t),$$

then equation (2.19) can be rewritten in the form

$$-\left(q|u'|^{p-2}\,u'\right)' = r\,h(u) \qquad \text{on } [a,b], \tag{2.20}$$

where q and r satisfy conditions (2.8), (2.9) and (2.10). By Claim 1, for any $c > s_0$ there is $d > c$ and a solution u of class C^2 of (2.20) which satisfy (2.12). Hence, u is also a solution of (2.19). Let us set

$$\beta(x_1,\dots,x_N) := u(x_1) \quad \text{for } (x_1,\dots,x_N) \in \overline{\Omega}.$$

Clearly, $\beta \in C^2(\overline{\Omega})$ and satisfies

$$c \le \beta(x_1,\dots,x_N) \le d \quad \text{and} \quad |\nabla\beta(x_1,\dots,x_N)| = |u'(x_1)| \ge 1 \quad \text{in } \overline{\Omega}.$$

Therefore, using (i_1), (2.3) and (2.12), we get

$$\begin{aligned}
-\operatorname{div} a(x,\nabla\beta(x)) &= -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x_1,\dots,x_N,u'(x_1),0,\dots,0)\\
&= -\frac{\partial a_1}{\partial \xi_1}(x_1,\dots,x_N,u'(x_1),0,\dots,0)\, u''(x_1)\\
&\quad -\sum_{i=1}^{N} \frac{\partial a_i}{\partial x_i}(x_1,\dots,x_N,u'(x_1),0,\dots,0)\\
&\ge -\gamma_1(\kappa + |u'(x_1)|)^{p-2}u''(x_1) - \gamma_4|u'(x_1)|^{p-1}\\
&\ge -\frac{\gamma_1}{2}|u'(x_1)|^{p-2}u''(x_1) + \gamma_4|u'(x_1)|^{p-2}u'(x_1)\\
&= h(u(x_1)) \;=\; h(\beta(x)).
\end{aligned}$$

This shows that β is a (classical) upper solution of problem (2.4). Thus, the proof of Claim 2 is concluded.

Finally, using Claim 2, one can build the required sequence of upper solutions $(\beta_n)_n$ of problem (2.4) satisfying condition (2.6). ∎

In a completely similar way, we can prove the following

Lemma 2.2 *Assume that*

$$\liminf_{s\to-\infty} \frac{H(s)}{s^p} \le 0.$$

Then, problem (2.4) *has a sequence* $(\alpha_n)_n$ *of lower solutions, with* $\alpha_n \in C^2(\overline{\Omega})$ *and*

$$\max_{\overline{\Omega}} \alpha_{n+1} < \min_{\overline{\Omega}} \alpha_n \to -\infty. \tag{2.21}$$

Now, we discuss the solvability of problem (2.4).

Lemma 2.3 *Assume that problem* (2.4) *admits a lower solution* α *and a sequence of upper solutions* $(\beta_n)_n$ *satisfying* (2.6). *Moreover, suppose that*

$$-\infty < \liminf_{s\to+\infty} \frac{H(s)}{s^p} \le \limsup_{s\to+\infty} \frac{H(s)}{s^p} = +\infty. \tag{2.22}$$

Then, problem (2.4) *has a sequence* $(z_n)_n$ *of solutions belonging to* $C^{1,\sigma}(\overline{\Omega})$*, for some* $\sigma > 0$*, and satisfying*

$$\alpha \leq z_1 \leq \ldots \leq z_n \leq \ldots \qquad \text{in } \overline{\Omega} \tag{2.23}$$

and

$$\max_{\overline{\Omega}} z_n \to +\infty. \tag{2.24}$$

Proof of Lemma 3. We closely follow an argument introduced in [13]. Let us define the functional

$$\phi : W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \to \mathbb{R}$$

by setting

$$\phi(w) = \int_\Omega A(x, \nabla w)\, dx - \int_\Omega H(w)\, dx.$$

Claim 1 *Assume that there exists a lower solution* α *and an upper solution* β *of problem* (2.4)*, satisfying* $\alpha \leq \beta$ *in* Ω*. Then, problem* (2.4) *has a solution* z *belonging to* $C^{1,\sigma}(\overline{\Omega})$*, for some* $\sigma > 0$*, such that*

$$\alpha \leq z \leq \beta \quad \text{in } \Omega \qquad \text{and} \qquad \phi(z) = \min_{\substack{w \in W_0^{1,p}(\Omega) \\ \alpha \leq w \leq \beta}} \phi(w).$$

The proof of Claim 1 employs a standard argument based on the minimization of the functional associated with a truncated equation. In particular, the observations made at the beginning of this section yield the weak lower semicontinuity of the functional, the validity of a weak comparison principle and, by [11, Th. 1], the regularity of the solution.

Let us take now a non–empty open set Ω_0, with $\overline{\Omega}_0 \subset \Omega$, and a function $\zeta \in C^1(\overline{\Omega})$ such that $\zeta(x) = 1$ on Ω_0, $\zeta(x) = 0$ and $\frac{\partial \zeta}{\partial \nu}(x) < 0$ on $\partial\Omega$, where ν is the outer normal to $\partial\Omega$. The proof of the following claim can be carried out exactly as in [13, Lemma 2.3].

Claim 2 *Assume* (2.22)*. Then, there is a sequence of real numbers* $(s_n)_n$*, with* $s_n \to +\infty$*, such that*

$$\phi(s_n \zeta) \to -\infty.$$

Finally, we are in position to build a sequence of solutions of (2.4). Take an upper solution, say β_1, such that $\beta_1 \geq \alpha$ in Ω. We get a solution z_1 in $C^{1,\sigma}(\overline{\Omega})$, for some $\sigma > 0$, of (2.4), with

$$\alpha \leq z_1 \leq \beta_1 \quad \text{in } \Omega \qquad \text{and} \qquad \phi(z_1) = \min_{\substack{w \in W_0^{1,p}(\Omega) \\ \alpha \leq w \leq \beta_1}} \phi(w).$$

Select a number, say s_1, such that

$$z_1 \le s_1\,\zeta \quad \text{in } \Omega \qquad \text{and} \qquad \phi(s_1\,\zeta) < \phi(z_1).$$

Take an upper solution, say β_2, such that $\beta_2 \ge s_1\,\zeta$ in Ω. We find a solution z_2 in $C^{1,\sigma}(\overline{\Omega})$ of (2.4), with

$$z_1 \le z_2 \le \beta_2 \quad \text{in } \Omega \qquad \text{and} \qquad \phi(z_2) = \min_{\substack{w\in W_0^{1,p}(\Omega)\\ z_1\le w\le \beta_2}} \phi(w).$$

Since $\phi(z_2) \le \phi(s_1\,\zeta) < \phi(z_1)$, we conclude that $z_1 \ne z_2$ and $\max_{\overline{\Omega}} z_2 > \min_{\overline{\Omega}} \beta_1$. Iterating this argument, we construct the required sequence of solutions of problem (2.4). ∎

In a similar way, we can prove the following:

Lemma 2.4 *Assume that problem* (2.4) *admits an upper solution β and a sequence of lower solutions $(\alpha_n)_n$ satisfying* (2.21)*. Moreover, suppose that*

$$-\infty < \liminf_{s\to-\infty} \frac{H(s)}{s^p} \le \limsup_{s\to-\infty} \frac{H(s)}{s^p} = +\infty.$$

Then, problem (2.4) *has a sequence $(y_n)_n$ of solutions, belonging to $C^{1,\sigma}(\overline{\Omega})$, for some $\sigma > 0$, and satisfying*

$$\ldots \le y_n \le \ldots \le y_1 \le \beta \qquad \textit{in } \overline{\Omega}$$

and

$$\min_{\overline{\Omega}} y_n \to -\infty.$$

Proof of Theorem 1.1. Let us consider the following comparison problems

$$\begin{cases} -\operatorname{div} a(x,\nabla u) = f(u) & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.25}$$

and

$$\begin{cases} -\operatorname{div} a(x,\nabla u) = g(u) & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.26}$$

¿From (j_1), using Lemma 2.1 with $h = g$, we deduce the existence of a sequence $(\beta_n)_n$ of upper solutions of problem (2.26) satisfying (2.6). It is clear that each β_n is also an upper solution of problem (2.25). From (j_3), using Lemma 2.2 with $h = f$, we deduce the existence of a sequence $(\alpha_n)_n$ of lower solutions of problem (2.25) satisfying (2.21). ¿From (j_2), using Lemma 2.3 with $h = f$, we deduce the existence of a sequence $(z_n)_n$ of solutions of problem (2.25), satisfying (2.23) and (2.24). Let us set, for each n,

$$\hat{\beta}_n := \beta_n \quad \text{and} \quad \hat{\alpha}_n := z_n.$$

It is clear that $\hat{\beta}_n$ and $\hat{\alpha}_n$ are, respectively, an upper solution and a lower solution of problem (1.1). Moreover, possibly passing to subsequences, we can suppose that

$$\max_{\overline{\Omega}} \hat{\alpha}_n < \min_{\overline{\Omega}} \hat{\beta}_n < \max_{\overline{\Omega}} \hat{\alpha}_{n+1} < \min_{\overline{\Omega}} \hat{\beta}_{n+1},$$

with $\max_{\overline{\Omega}} \hat{\alpha}_n \to +\infty$. Hence, by [5], we find for each n a solution $\hat{u}_n$ of problem (1.1) such that

$$\hat{\alpha}_n \leq \hat{u}_n \leq \hat{\beta}_n \quad \text{in } \Omega.$$

Now, set $u_1 := \hat{u}_1$. From [3, Lemma 3.1 and Remark 3.3], we have that, for each $n \geq 1$, there exists a solution u_{n+1} of (1.1) such that

$$\max\{u_n, \hat{u}_{n+1}\} \leq u_{n+1} \leq \hat{\beta}_{n+1} \quad \text{in } \Omega.$$

Using this fact, we can finally build a sequence $(u_n)_n$ of solutions of problem (1.1) satisfying

$$u_1 \leq \ldots \leq u_n \leq u_{n+1} \leq \ldots \quad \text{in } \Omega \qquad \text{and} \qquad \max_{\overline{\Omega}} u_n \to +\infty.$$

In a completely similar way, we construct a sequence $(v_n)_n$ of solutions of problem (1.1) satisfying

$$u_1 \geq v_1 \geq \ldots \geq v_n \geq v_{n+1} \geq \ldots \quad \text{in } \Omega \qquad \text{and} \qquad \min_{\overline{\Omega}} v_n \to -\infty.$$

Hence, Theorem 1.1 is proved. ∎

Proof of Theorem 1.2. We proceed exactly as in the proof of Theorem 1.1, observing that any lower solution of the elliptic problem (2.25) is a lower solution of the parabolic problem (1.2) and any upper solution of (2.26) is an upper solution of (1.2). Of course, here we have to use [6] instead of [5].

Regarding the ordering of the solutions, we can exploit, when $p \geq 2$, a parabolic counterpart of the result in [3]. This statement can be proved by a modification (as suggested in [2]) of the argument produced in [1] for the initial value problem. Another proof can be found in [10]. ∎

Proof of Proposition 1.1. The preceding argument yields the conclusion, as soon as one observes that $\alpha = 0$ is a lower solution. ∎

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