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GENERALIZED UNIFORMLY CONTINUOUS SOLUTION OPERATORS AND INHOMOGENEOUS FRACTIONAL EVOLUTION EQUATIONS WITH VARIABLE COEFFICIENTS

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Abstract. We consider Cauchy problem for inhomogeneous fractional evolution equations with Caputo fractional derivatives of order $0 < \alpha < 1$ and variable coefficients depending on x. In order to solve this problem we introduce generalized uniformly continuous solution operators and use them to obtain the unique solution on a certain Colombeau space. In our solving procedure, instead of the original problem we solve a certain approximate problem, but therefore we also prove that the solutions of these two problems are associated. At the end, we illustrate the applications of the developed theory by giving some appropriate examples.

1. Introduction

Fractional evolution equations have been studied very often in the previous decades because of their numerous applications. Many well known problems are, in fact special cases of fractional evolution equations. For instance, both time fractional diffusion problem and time fractional reaction-advection-diffusion problem are of that type. In literature, authors have mainly considered several cases: homogeneous case, case when f is linear or case with constant coefficients. In this paper, we want to study semilinear problem which also includes space variable coefficients, i.e. the equation of the type

$$ {}^{C}\mathcal{D}_t^{\alpha} u(t) = Au(t) + f(\cdot, t, u), \qquad u(0) = u_0, \tag{1.1} $$

where ${}^{C}\mathcal{D}_t^{\alpha}$ is the Caputo's fractional derivative of order $0 < \alpha < 1$ and A is a linear, closed operator densely defined on some Banach space.

Semilinear fractional Cauchy problem with variable coefficients in general case has been solved approximately, usually applying different numerical methods. One of the reasons why we have considered fractional equations in the framework of the Colombeau theory of generalized functions is the intention that these equations be treated using operator's approach, that is, applying the solution operators as generalization of semigroup of operators.

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In our solving procedure, instead of the original problem (1.1) we consider the approximate problem

$$ {}^{C}\mathcal{D}_t^{\alpha} u(t) = \widetilde{A}u(t) + f(\cdot, t, u), \qquad u(0) = u_0, \tag{1.2} $$

where $\widetilde{A}$ is a generalized linear bounded operator associated (in certain sense) to the original operator A. Therefore we will pay special attention to comparison analysis of these two problems. To solve the approximate problem, we introduce a notion of generalized uniformly continuous solution operator generated by $\widetilde{A}$. This generalized solutions operator is, in fact, a generalization of generalized uniformly continuous semigroup of operators. (For $\alpha = 1$ a generalized solution operator is defined as a generalized semigroup of operators). Generalized uniformly continuous semigroups were introduced in [18] and the theory has been developed later in [19] in order to use the theory of semigroups in solving some partial differential equations with singularities in some generalized function spaces.

Solution operators as a generalization of C_0 semigroups and cosine families of operators are introduced by Bazhlekova in [5]. In [4] and [5] the corresponding solution operator theory was developed for solving some homogeneous fractional evolution problems. We remark that in some literature the solution operator is also called fractional resolvent family or fractional resolvent operator function (see e.g. [7, 16]).

In this article, we solve problem (1.2) in the framework of the Colombeau theory. The theory of Colombeau generalized functions is developed in order to make possible studying some nonlinear differential equations that can not be treated neither classically (there is no classical solution) nor in distributional sense (nonlinear problems include the multiplication and the multiplication of distribution is not well defined). For the Colombeau theory in general we refer, for example, to [6, 8, 17, 20].

In [14] we considered a special case of (1.2) for $\alpha = 1$ and with Colombeau generalized operator $\widetilde{A}$ defined by space fractional derivatives. In this paper we make a step further by considering the problem with fractional time derivative of order $0 < \alpha < 1$. We obtain the unique solution to the problem (1.2) in a certain Colombeau space. In case when A is a differential operator (integer of fractional order) the regularization is necessary in order to obtain bounded operators. Our method admits variable coefficients in both $\tilde{A}$ and A.

This article is organized as follows. Fractional derivatives and some useful estimates involving them are investigated in Section 2. A part of this section is devoted to the Mittag-Leffler function since it has an important role in defining the solution operator. Colombeau spaces that we use later in the paper are defined in Section 3. In Section 4 we define uniformly continuous solution operators and prove some basic properties. In Section 5 we introduce the Colombeau uniformly continuous solution operators and develop the corresponding theory. After setting the framework theory, in Section 6 we investigate the inhomogeneous problem (1.2). We prove that the problem has a unique solution in a certain Colombeau space. Since in the whole paper, instead of the original problem (1.1) we study the corresponding approximate problem (1.2), Section 7 is devoted to a comparison analysis of these two problems. Finally, in the last section we illustrate how one can use our theory in solving some fractional evolution problems appearing in applications, such as time and time-space fractional diffusion equation and also time-space fractional reaction-advection-diffusion equation. In these problems the corresponding differential operators will be in the form of regularized operators, in order to transform unbounded differential operators into (integral) bounded operators.

2. Time fractional derivatives and some useful estimates

In this section we recall definitions of fractional derivatives with respect to time variable and give some useful estimates for fractional derivatives and Mittag-Leffler function that we will use later.

2.1. **Fractional derivatives with respect to time variable.** The Caputo fractional derivative of order α, $m-1<\alpha\le m$, $m\in\mathbb{N}$, has the form (see, for example, [1, 2, 15, 23, 26])

$$ {}^{C}\mathcal{D}_t^{\alpha} f(t) = J_t^{m-\alpha} f^{(m)}(t), \tag{2.1}$$

where J_t^{α}, $\alpha\ge 0$, is a fractional integral for function $f(t)$ given by

$$ J_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau,$$

with $J^0=I$, I is identity operator.

The Riemann-Liouville fractional derivative of order α, $m-1<\alpha\le m$, $m\in\mathbb{N}$, is given by

$$ {}^{RL}\mathcal{D}_t^{\alpha} f(t) = \frac{d^m}{dt^m} J_t^{m-\alpha} f(t). \tag{2.2}$$

Recall that ${}^{C}\mathcal{D}_t^{\alpha}$ is a left inverse of J_t^{α}, i.e. ${}^{C}\mathcal{D}_t^{\alpha} J_t^{\alpha} f = f$ for a continuous function f, but in general it is not a right inverse [10, Theorems 3.7, 3.8]. In general, for an absolutely continuous function f and $0<\alpha<1$, the following holds $J_t^{\alpha}{}^{C}\mathcal{D}_t^{\alpha} f(t) = f(t)-f(0)$.

The following holds for a Riemann-Liouville fractional derivative:

Proposition 2.1 ([28, Lemma 2.1])**.** *For all $\alpha\in(m-1,m]$ and $\beta\ge 0$, it holds*

$$ J_t^{\beta+\alpha} f(t) = J_t^{\beta+m}\, {}^{RL}\mathcal{D}_t^{m-\alpha} f(t). \tag{2.3}$$

Remark 2.2. Taking into account the properties of Mittag-Leffler function and its integer order derivatives (especially at zero), the Colombeau space from which we choose generalized solution operators will be defined using the space $C^{m-1}([0,\infty):\mathcal{L}(E))\cap C^m((0,\infty):\mathcal{L}(E))$ and supposing some additional properties (see Definition 3.1 and Definition 3.2). It is the space of continuously differentiable functions with respect to t and with values in space $\mathcal{L}(E)$, where $(E,\|\cdot\|)$ is a Banach space and $\mathcal{L}(E)$ is the space of all linear continuous mappings from E into E with the norm

$$ \|A\|_{\mathcal{L}(E)} = \sup_{x\in E,\, x\neq 0} \frac{\|Ax\|_E}{\|x\|_E}.$$

The following lemma will play an important role in later proofs.

Lemma 2.3. *Let $(E,\|\cdot\|)$ be a Banach space and $\mathcal{L}(E)$ the space of all linear continuous mappings from E into E. Let $m-1<\alpha<m$, $m\in\mathbb{N}$. Suppose that $(\cdot,t)\to f(\cdot,t)\in C^{m-1}([0,\infty):\mathcal{L}(E))\cap C^m((0,\infty):\mathcal{L}(E))$ is such that $\lim_{t\to 0^+}\big\|\frac{f^{(m)}(\cdot,t)}{t^{\alpha-m}}\big\|_{\mathcal{L}(E)} = C<+\infty$. Then*

$$ {}^{C}\mathcal{D}_t^{\alpha} f(\cdot,t) = \lim_{\eta\to 0^+} {}^{C}_{\eta}\mathcal{D}_t^{\alpha} f(\cdot,t) \quad \text{in } \mathcal{L}(E), \tag{2.4}$$

where

$$ {}_{\eta}^{C}\mathcal{D}_{t}^{\alpha} f(\cdot,t) = \frac{1}{\Gamma(m-\alpha)} \int_{\eta}^{t} \frac{f^{(m)}(\cdot,\tau)}{(t-\tau)^{\alpha-m+1}} d\tau. \tag{2.5} $$

Proof. Fix $m \in \mathbb{N}$ and α such that $m-1 < \alpha < m$. Then

$$ \begin{aligned} &\|{}^{C}\mathcal{D}_{t}^{\alpha} f(\cdot,t) - {}_{\eta}^{C}\mathcal{D}_{t}^{\alpha} f(\cdot,t)\|_{\mathcal{L}(E)} \\ &\leq \frac{1}{\Gamma(m-\alpha)} \int_{0}^{\eta} \frac{\|f^{(m)}(\cdot,\tau)\|_{\mathcal{L}(E)}}{(t-\tau)^{\alpha-m+1}} d\tau \\ &= \frac{1}{\Gamma(m-\alpha)} \int_{0}^{\eta} \frac{\|f^{(m)}(\cdot,\tau)\|_{\mathcal{L}(E)}}{\tau^{\alpha-m}} \frac{\tau^{\alpha-m}}{(t-\tau)^{\alpha-m+1}} d\tau \\ &\leq \frac{1}{\Gamma(m-\alpha)} \sup_{\tau\in[0,\eta]} \|\frac{f^{(m)}(\cdot,\tau)}{\tau^{\alpha-m}}\|_{\mathcal{L}(E)} \int_{0}^{\eta} \frac{\tau^{\alpha-m}}{(t-\tau)^{\alpha-m+1}} d\tau \\ &\leq \frac{1}{\Gamma(m-\alpha)} \sup_{\tau\in[0,\eta]} \|\frac{f^{(m)}(\cdot,\tau)}{\tau^{\alpha-m}}\|_{\mathcal{L}(E)} \frac{\eta^{\alpha-m+1}}{(\alpha-m+1)(t-\eta)^{\alpha-m+1}}. \end{aligned} $$

Letting $\eta \to 0^{+}$ one easily gets (2.4). □

The following assertion is so-called fractional mean value theorem.

Theorem 2.4 ([21])**.** *Let $0 < \alpha < 1$. For $t \to f(t) \in C[a,b]$ and ${}_{a}^{C}\mathcal{D}_{t}^{\alpha} f \in C(a,b]$, the following holds*

$$ f(t) = f(a) + \frac{1}{\Gamma(1+\alpha)} ({}_{a}^{C}\mathcal{D}_{t}^{\alpha} f)(\xi)(t-a)^{\alpha}, \quad a \leq \xi \leq t,\ t \in (a,b], $$

where ${}_{a}^{C}\mathcal{D}_{t}^{\alpha} f$ is defined as in (2.5).

2.2. **Mittag-Leffler function.** The two-parameter Mittag-Leffler function $E_{\alpha,\beta}$ is given by

$$ E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^{n}}{\Gamma(\beta+n\alpha)}, \quad z \in \mathbb{C},\ \alpha > 0,\ \beta \in \mathbb{C}. $$

When $\beta = 1$ we shortly write $E_{\alpha,1}(z) \equiv E_{\alpha}(z)$.

If $0 < \alpha < 2$ and $\beta > 0$ then, for $|z| \to \infty$,

$$ E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} \exp(z^{1/\alpha}) + \varepsilon_{\alpha,\beta}(z), \quad |arg z| \leq \frac{\alpha\pi}{2}, \tag{2.6} $$

where

$$ \varepsilon_{\alpha,\beta}(z) = -\sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(\beta-\alpha n)} + \mathcal{O}(|z|^{-N}), $$

for some $N \in \mathbb{N}$, $N \neq 1$ (see [9]).

Using the previous asymptotic expansion when $|z| \to \infty$, one can get a very useful estimation for the two-parameter Mittag-Leffler function.

Proposition 2.5. *Let $0 < \alpha < 2$ and $\beta > 0$. Then*

$$ E_{\alpha,\beta}(\omega t^{\alpha}) \leq C_{\alpha,\beta}(1+\omega^{(1-\beta)/\alpha})(1+t^{1-\beta}) \exp(\omega^{1/\alpha} t), \quad \omega \geq 0,\ t \geq 0. \tag{2.7} $$

Proof. For $\omega = 0$ and all $t \geq 0$, the inequality is trivially satisfied. Fix $0 < \alpha < 2$, $\beta > 0$ and $\omega > 0$. Choose an arbitrarily large $T > 0$. Then from (2.6), for all $t > (\frac{T}{\omega})^{\frac{1}{\alpha}}$, follows that there exists a constant $C_1 > 0$ such that

$$ E_{\alpha,\beta}(\omega t^{\alpha}) \leq C_1 (\omega t^{\alpha})^{(1-\beta)/\alpha} \exp\big((\omega t^{\alpha})^{1/\alpha}\big) $$

$$\begin{aligned} &= C_1\omega^{(1-\beta)/\alpha}t^{1-\beta}\exp(\omega^{1/\alpha}t) \\ &\le C_1(1+\omega^{(1-\beta)/\alpha})(1+t^{1-\beta})\exp(\omega^{1/\alpha}t). \end{aligned}$$

Since $E_{\alpha,\beta}$ is a continuous function, for all $t \in [0,\infty)$, we have that, for $t \in [0,(\frac{T}{\omega})^{\frac{1}{\alpha}}]$, there exists a constant C_2 such that

$$E_{\alpha,\beta}(\omega t^\alpha) \le C_2 \le C_2(1+\omega^{(1-\beta)/\alpha})(1+t^{1-\beta})\exp(\omega^{1/\alpha}t).$$

Taking $C_{\alpha,\beta} = \max\{C_1, C_2\}$ we obtain the inequality (2.7). □

The linear Cauchy problem (1.1) with Caputo fractional derivatives has been considered in [28], but in some special spaces of L^p functions whose Fourier transforms are compactly supported in a some domain G, and the following result was obtained.

Proposition 2.6 (Fractional Duhamel principle [28])**.** *The solution of the Cauchy problem* (1.1) *is given by*

$$u(t) = E_\alpha(t^\alpha A)u_0 + \int_0^t E_\alpha((t-\tau)^\alpha A)\, {}^{RL}\mathcal{D}_\tau^{1-\alpha} f(\tau)d\tau. \tag{2.8}$$

3. Colombeau spaces

Let $(E, \|\cdot\|)$ be a Banach space and $\mathcal{L}(E)$ the space of all linear continuous mappings from E into E.

Definition 3.1. Let $m-1<\alpha<m$, $m\in\mathbb{N}$. $\mathcal{S}E_M^{\alpha,m}([0,\infty):\mathcal{L}(E))$ is the space of nets

$$(S_\alpha)_\varepsilon : [0,\infty) \to \mathcal{L}(E), \quad \varepsilon \in (0,1),$$

with the following properties:

(i) $t \to (S_\alpha)_\varepsilon(t) \in C^{m-1}([0,\infty):\mathcal{L}(E)) \cap C^m((0,\infty):\mathcal{L}(E))$.

(ii) $\lim_{t\to 0^+} \|\frac{\frac{d^m}{dt^m}(S_\alpha)_\varepsilon(t)}{t^{\alpha-m}}\|_{\mathcal{L}(E)} = C < +\infty$.

(iii) For every $T>0$ there exist $N\in\mathbb{N}$, $M>0$ and $\varepsilon_0\in(0,1)$ such that

$$\sup_{t\in[0,T)} \|{}^C\mathcal{D}_t^\gamma (S_\alpha)_\varepsilon(t)\|_{\mathcal{L}(E)} \le M\varepsilon^{-N}, \quad \varepsilon<\varepsilon_0,\ \gamma\in\{0,\dots,m-1,\alpha\},$$

$$\sup_{t\in(0,T)} \|\frac{d^m}{dt^m}(S_\alpha)_\varepsilon(t)\|_{\mathcal{L}(E)} \le M\varepsilon^{-N}, \quad \varepsilon<\varepsilon_0.$$

Similarly we define the following space.

Definition 3.2. Let $m-1<\alpha<m$, $m\in\mathbb{N}$. $\mathcal{S}N_{\alpha,m}([0,\infty):\mathcal{L}(E))$ is the space of nets

$$(N_\alpha)_\varepsilon : [0,\infty) \to \mathcal{L}(E), \quad \varepsilon \in (0,1)$$

with the following properties:

(i) $t \to (N_\alpha)_\varepsilon(t) \in C^{m-1}([0,\infty):\mathcal{L}(E)) \cap C^m((0,\infty):\mathcal{L}(E))$.

(ii) $\lim_{t\to 0^+} \|\frac{\frac{d^m}{dt^m}(N_\alpha)_\varepsilon(t)}{t^{\alpha-m}}\|_{\mathcal{L}(E)} = C < +\infty$.

(iii) For every $T>0$ and $a\in\mathbb{R}$ there exist $M>0$ and $\varepsilon_0\in(0,1)$ such that

$$\begin{gathered} \sup_{t\in[0,T)} \|{}^C\mathcal{D}_t^\gamma (N_\alpha)_\varepsilon(t)\|_{\mathcal{L}(E)} \le M\varepsilon^{a}, \quad \varepsilon<\varepsilon_0,\ \gamma\in\{0,\dots,m-1,\alpha\}, \\ \sup_{t\in(0,T)} \|\frac{d^m}{dt^m}(N_\alpha)_\varepsilon(t)\|_{\mathcal{L}(E)} \le M\varepsilon^{a}, \quad \varepsilon<\varepsilon_0. \end{gathered} \tag{3.1}$$

When $m = 1$ we denote

$$\mathcal{S}E_M^{\alpha,1}([0,\infty) : \mathcal{L}(E)) = \mathcal{S}E_M^{\alpha}([0,\infty) : \mathcal{L}(E)),$$
$$\mathcal{S}N_{\alpha,1}([0,\infty) : \mathcal{L}(E)) = \mathcal{S}N_{\alpha}([0,\infty) : \mathcal{L}(E)).$$

In this article, Caputo's fractional derivative in the problem of consideration is of order $0 < \alpha < 1$ and therefore, from now on, we will consider only that case. Hence, further we investigate spaces $\mathcal{S}E_M^{\alpha}([0,\infty) : \mathcal{L}(E))$ and $\mathcal{S}N_{\alpha}([0,\infty) : \mathcal{L}(E))$, although all the assertions we give can be extended for all $m \in \mathbb{N}$.

Proposition 3.3. *The space $\mathcal{S}E_M^{\alpha}([0,\infty) : \mathcal{L}(E))$ is an algebra with respect to composition of operators, and $\mathcal{S}N_{\alpha}([0,\infty) : \mathcal{L}(E))$ is an ideal of $\mathcal{S}E_M^{\alpha}([0,\infty) : \mathcal{L}(E))$.*

Proof. Fix $0 < \alpha < 1$ and let S_α and T_α are from the space $\mathcal{S}E_M^{\alpha}([0,\infty) : \mathcal{L}(E))$. Then, it easily follows that $S_\alpha(t)T_\alpha(t)$ satisfies the properties (i) and (ii) from Definition (3.1). The fact that $S_\alpha(t)T_\alpha(t)$ satisfies the property (iii) for $\gamma \in \{0,1\}$, can be proved in the usual way as in the case of Colombeau spaces with integer order derivatives.

Let us prove that (iii) is satisfied for $\gamma = \alpha$, too. Indeed, for $t \in (\eta, T)$, where $\eta > 0$ is arbitrarily small and $T > 0$, using the property (2.4) we have

$$\begin{aligned}
&\|{}^C\mathcal{D}_t^{\alpha}((S_\alpha)_\varepsilon(t)(T_\alpha)_\varepsilon(t))\|_{\mathcal{L}(E)}\\
&\le \frac{1}{\Gamma(1-\alpha)} \lim_{\eta\to 0^+} \int_\eta^t \frac{\|((S_\alpha)_\varepsilon(\tau)(T_\alpha)_\varepsilon(\tau))'\|_{\mathcal{L}(E)}}{(t-\tau)^\alpha} d\tau\\
&\le \frac{1}{\Gamma(1-\alpha)} \lim_{\eta\to 0^+} \int_\eta^t \frac{\|((S_\alpha)_\varepsilon(\tau))'\|_{\mathcal{L}(E)}\|(T_\alpha)_\varepsilon(\tau)\|_{\mathcal{L}(E)}}{(t-\tau)^\alpha} d\tau\\
&\quad + \frac{1}{\Gamma(1-\alpha)} \lim_{\eta\to 0^+} \int_\eta^t \frac{\|(S_\alpha)_\varepsilon(\tau)\|_{\mathcal{L}(E)}\|((T_\alpha)_\varepsilon(\tau))'\|_{\mathcal{L}(E)}}{(t-\tau)^\alpha} d\tau\\
&\le \lim_{\eta\to 0^+} \frac{(t-\eta)^{1-\alpha}}{\Gamma(2-\alpha)} M_1\varepsilon^{-N}\\
&\le \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} M_1\varepsilon^{-N}.
\end{aligned}$$

Thus, we obtain the moderate bound for $t \in (0,T)$, i.e.

$$\sup_{t\in(0,T)} \|{}^C\mathcal{D}_t^{\alpha}((S_\alpha)_\varepsilon(t)(T_\alpha)_\varepsilon(t))\|_{\mathcal{L}(E)} \le M\varepsilon^{-N}.$$

It remains to prove the moderate bound for $\|{}^C\mathcal{D}_t^{\alpha}((S_\alpha)_\varepsilon(t)(T_\alpha)_\varepsilon(t))\big|_{t=0}\|$. From Theorem 2.4 we obtain

$$\begin{aligned}
&{}^C\mathcal{D}_t^{\alpha}((S_\alpha)_\varepsilon(t)(T_\alpha)_\varepsilon(t))\big|_{t=0}\\
&= \frac{1}{\Gamma(1+\alpha)} \lim_{t\to 0^+} \frac{(S_\alpha)_\varepsilon(t)(T_\alpha)_\varepsilon(t) - (S_\alpha)_\varepsilon(0)(T_\alpha)_\varepsilon(0)}{t^\alpha}\\
&= \frac{1}{\Gamma(1+\alpha)} \lim_{t\to 0^+} \frac{((S_\alpha)_\varepsilon(t) - (S_\alpha)_\varepsilon(0))(T_\alpha)_\varepsilon(t)}{t^\alpha}\\
&\quad + \frac{1}{\Gamma(1+\alpha)} \lim_{t\to 0^+} \frac{(S_\alpha)_\varepsilon(0))((T_\alpha)_\varepsilon(t) - (T_\alpha)_\varepsilon(0))}{t^\alpha}\\
&= {}^C\mathcal{D}_t^{\alpha}((S_\alpha)_\varepsilon(t))\big|_{t=0}(T_\alpha)_\varepsilon(0) + (S_\alpha)_\varepsilon(0)\,{}^C\mathcal{D}_t^{\alpha}((T_\alpha)_\varepsilon(t))\big|_{t=0}.
\end{aligned}$$

Estimating in norm, we obtain a moderate bound for $\|{}^C\mathcal{D}_t^\alpha((S_\alpha)_\varepsilon(t)(T_\alpha)_\varepsilon(t))\big|_{t=0}\|$. Thus, (iii) is satisfied.

Similarly, one can prove that $(T_\alpha)_\varepsilon(t)(S_\alpha)_\varepsilon(t)$, also satisfies all properties from Definition (3.1). Thus, the space $\mathcal{S}E_M^\alpha([0,\infty):\mathcal{L}(E))$ is an algebra. One can similarly prove that the space $\mathcal{S}N_\alpha([0,\infty):\mathcal{L}(E))$ is an ideal of $\mathcal{S}E_M^\alpha([0,\infty):\mathcal{L}(E))$. □

Now we can define a Colombeau-type space as a factor algebra by

$$\mathcal{S}G_\alpha([0,\infty):\mathcal{L}(E))=\frac{\mathcal{S}E_M^\alpha([0,\infty):\mathcal{L}(E))}{\mathcal{S}N_\alpha([0,\infty):\mathcal{L}(E))}. \tag{3.2}$$

For every $0<\alpha<1$ elements of $\mathcal{S}G_\alpha([0,\infty):\mathcal{L}(E))$ will be denoted by $S=[(S_\alpha)_\varepsilon]$, where $(S_\alpha)_\varepsilon$ is a representative of the class.

Similarly, one can define the following spaces: $\mathcal{S}E_M(E)$ is the space of nets of linear continuous mappings

$$A_\varepsilon: E\to E,\quad \varepsilon\in(0,1),$$

with the property that there exists constants $N\in\mathbb{N}$, $M>0$ and $\varepsilon_0\in(0,1)$ such that

$$\|A_\varepsilon\|_{\mathcal{L}(E)}\le M\varepsilon^{-N},\quad \varepsilon<\varepsilon_0.$$

$\mathcal{S}N(E)$ is the space of nets of linear continuous mappings $A_\varepsilon: E\to E$, $\varepsilon\in(0,1)$, with the property that for every $a\in\mathbb{R}$, there exist $M>0$ and $\varepsilon_0\in(0,1)$ such that

$$\|A_\varepsilon\|_{\mathcal{L}(E)}\le M\varepsilon^{a},\quad \varepsilon<\varepsilon_0.$$

The Colombeau space of generalized linear operators on E is defined by

$$\mathcal{S}G(E)=\frac{\mathcal{S}E_M(E)}{\mathcal{S}N(E)}.$$

Elements of $\mathcal{S}G(E)$ will be denoted by $A=[A_\varepsilon]$, where A_ε is a representative of the class.

Finally, we introduce the Colombeau space within which we will solve (1.2). We give the definitions for arbitrary $m\in\mathbb{N}$. Let $m-1<\alpha<m$, $m\in\mathbb{N}$. $\mathcal{E}_M^\alpha([0,\infty):H^m(\mathbb{R}))$ is the space of nets

$$G_\varepsilon:[0,\infty)\times\mathbb{R}\to\mathbb{C},\quad \varepsilon\in(0,1),$$

with the following properties:

(i) $G_\varepsilon(\cdot,\cdot)\in C^{m-1}([0,\infty):H^m(\mathbb{R}))\cap C^m((0,\infty):H^m(\mathbb{R}))$.

(ii) $\lim_{t\to0^+}\|\frac{\frac{d^m}{dt^m}G_\varepsilon(t,\cdot)}{t^{\alpha-m}}\|_{H^m}=C<+\infty$.

(iii) For every $T>0$ there exist $M>0, N\in\mathbb{N}$ and $\varepsilon_0>0$ such that

$$\begin{gathered}\sup_{t\in[0,T)}\|{}^C\mathcal{D}_t^\gamma G_\varepsilon(t,\cdot)\|_{H^m}\le M\varepsilon^{-N},\quad \varepsilon<\varepsilon_0,\ \gamma\in\{0,\dots,m-1,\alpha\},\\ \sup_{t\in(0,T)}\|\frac{d^m}{dt^m}G_\varepsilon(t,\cdot)\|_{H^m}\le M\varepsilon^{-N},\quad \varepsilon<\varepsilon_0.\end{gathered} \tag{3.3}$$

It is an algebra with respect to multiplication.

Similarly, for $m-1<\alpha<m$, $m\in\mathbb{N}$, $\mathcal{N}_\alpha([0,\infty):H^m(\mathbb{R}))$ is the space of nets $G_\varepsilon\in\mathcal{E}_M^\alpha([0,\infty):H^m(\mathbb{R}))$ with the following properties:

(i) $G_\varepsilon(\cdot,\cdot)\in C^{m-1}([0,\infty):H^m(\mathbb{R}))\cap C^m((0,\infty):H^m(\mathbb{R}))$.

(ii) $\lim_{t\to0^+}\|\frac{\frac{d^m}{dt^m}G_\varepsilon(t,\cdot)}{t^{\alpha-m}}\|_{H^m}=C<+\infty$.

(iii) For every $T > 0$ and $a \in \mathbb{R}$ there exist $M > 0$ and $\varepsilon_0 > 0$ such that

$$\begin{gathered} \sup_{t\in[0,T)} \|{}^C\mathcal{D}_t^\gamma G_\varepsilon(t,\cdot)\|_{H^m} \le M\varepsilon^a, \quad \varepsilon < \varepsilon_0,\ \gamma \in \{0,\dots,m-1,\alpha\}, \\ \sup_{t\in(0,T)} \|\frac{d^m}{dt^m} G_\varepsilon(t,\cdot)\|_{H^m} \le M\varepsilon^a, \quad \varepsilon < \varepsilon_0. \end{gathered} \tag{3.4}$$

The space $\mathcal{N}_\alpha([0,\infty) : H^m(\mathbb{R}))$ is an ideal of $\mathcal{E}_M^\alpha([0,\infty) : H^m(\mathbb{R}))$.

The quotient space

$$\mathcal{G}_\alpha([0,\infty) : H^m(\mathbb{R})) = \frac{\mathcal{E}_M^\alpha([0,\infty) : H^m(\mathbb{R}))}{\mathcal{N}_\alpha([0,\infty) : H^m(\mathbb{R}))}$$

is the corresponding Colombeau generalized function space related to the Sobolev space H^m. Again, in this paper we will consider only the case $m = 1$ and $m = 2$, i.e. the solution of our fractional evolution problem will be an element of $\mathcal{G}_\alpha([0,\infty) : H^1(\mathbb{R}))$ or $\mathcal{G}_\alpha([0,\infty) : H^2(\mathbb{R}))$.

In a similar way, by omitting variable t, one can define spaces $\mathcal{E}_M^\alpha(H^m(\mathbb{R}))$, $\mathcal{N}_\alpha(H^m(\mathbb{R}))$, and $\mathcal{G}_\alpha(H^m(\mathbb{R}))$.

4. Uniformly continuous solution operators

Consider the Cauchy problem for the fractional evolution equation of order α with $0 < \alpha < 1$,

$${}^C\mathcal{D}_t^\alpha u(t) = Au(t), \quad t > 0; \quad u(0) = x, \tag{4.1}$$

where ${}^C\mathcal{D}_t^\alpha$ is the Caputo fractional derivative of order α, and A is a linear and bounded operator defined on a Banach space E. The more general case when A is a closed linear operator densely defined in a Banach space E was considered in [5]. As it is pointed out in [5], the problem (4.1) is well-posed if and only if the Volterra integral equation

$$u(t) = x + \int_0^t g_\alpha(t-\tau)Au(\tau)d\tau \tag{4.2}$$

is well-possed, where $g_\alpha(t)$ is defined for $\alpha > 0$, by

$$g_\alpha(t) = \begin{cases} t^{\alpha-1}/\Gamma(\alpha), & t > 0, \\ 0, & t \le 0. \end{cases}$$

In the general case when A is a closed linear operator densely defined in a Banach space E, strongly continuous solution operator for (4.1) is introduced in [5]. Similarly, when A is linear and bounded, we introduce uniformly continuous solution operator.

Definition 4.1. A family $S_\alpha(t),\ t \ge 0$, of linear and bounded operators on Banach space E is called a uniformly continuous solution operator for (4.1) if the following conditions are satisfied:

(i) $S_\alpha(t)$ is a uniformly continuous function for $t \ge 0$ and $S_\alpha(0) = I$, where I is identity operator on E.
(ii) $AS_\alpha(t)x = S_\alpha(t)Ax$, for all $x \in E$, $t \ge 0$.
(iii) $S_\alpha(t)x$ is a solution of (4.2) for all $x \in E$, $t \ge 0$.

Definition 4.2. The infinitesimal generator A of a uniformly continuous solution operator $S_\alpha(t)$, $\alpha > 0$, $t \geq 0$, for (4.1) is defined by

$$Ax = \Gamma(1+\alpha) \lim_{t \downarrow 0} \frac{S_\alpha(t)x - x}{t^\alpha}, \tag{4.3}$$

for all $x \in E$.

The generator A could also be defined as

$$Ax = ({}^C\mathcal{D}_t^\alpha S_\alpha)(t)x\big|_{t=0},$$

since $J_t^\alpha {}^C\mathcal{D}_t^\alpha S_\alpha(t)x = S_\alpha(t)x - x$ and for all functions $v \in C(\mathbb{R}_+; E)$ holds

$$\lim_{t \downarrow 0} \frac{J_t^\alpha v(t)}{g_{\alpha+1}(t)} = v(0)$$

(see [5]).

Remark 4.3. In the case $0 < \alpha \leq 1$, the definition given by (4.3) also follows from Theorem 2.4.

Definition 4.4 ([5])**.** The solution operator $S_\alpha(t)$ is called exponentially bounded if there exist constants $M \geq 1$ and $\omega \geq 0$ such that

$$\|S_\alpha(t)\| \leq Me^{\omega t}, \ t \geq 0.$$

Theorem 4.5 ([5, Theorem 2.5])**.** *Let $\alpha > 0$. Then exponentially bounded uniformly continuous solution operator $S_\alpha(t)$ is the solution operator for the Cauchy problem* (4.1) *if and only if $A \in \mathcal{L}(E)$.*

From Definition 4.2 it follows that every solution operator has a unique infinitesimal generator. If $S_\alpha(t)$ is a uniformly continuous solution operator satisfying $\|S_\alpha(t)\| \leq Me^{\omega t}$, for some $M \geq 1$ and $\omega \geq 0$, its infinitesimal generator is a bounded linear operator.

On the other hand, every bounded linear operator A is the infinitesimal generator of a uniformly continuous solution operator given by

$$S_\alpha(t) = E_\alpha(t^\alpha A) = \sum_{n=0}^{\infty} \frac{t^{n\alpha} A^n}{\Gamma(1+n\alpha)}, \quad \alpha > 0, \ t \geq 0.$$

For every $0 < \alpha \leq 1$ this solution operator is unique as asserted in the following theorem.

Theorem 4.6. *Let $0 < \alpha \leq 1$ and let $S_\alpha(t)$ and $T_\alpha(t)$ be exponential bounded uniformly continuous solution operators with infinitesimal generators A and B, respectively. If $A = B$ then $S_\alpha(t) = T_\alpha(t)$, for every $t \geq 0$.*

Proof. Since $S_\alpha(t)$ is exponential bounded there exist constants $M \geq 1$ and $\omega_1 \geq 0$ such that

$$\|S_\alpha(t)\| \leq Me^{\omega_1 t}, \ t \geq 0.$$

Then for $Re\lambda > \omega_1$ and $x \in E$ we have

$$\lambda^{\alpha-1} R(\lambda^\alpha, A)x = \int_0^\infty e^{-\lambda t} S_\alpha(t)x dt,$$

where $R(\lambda, A) = (\lambda I - A)^{-1}$ stands for the resolvent operator of A. Similarly, for $T_\alpha(t)$ there exists $\omega_2 \ge 0$ such that for $Re\lambda > \omega_2$ and $x \in E$ we have

$$\lambda^{\alpha-1} R(\lambda^\alpha, A)x = \int_0^\infty e^{-\lambda t} T_\alpha(t) x dt,$$

and $S_\alpha(t) = T_\alpha(t)$ follows from the uniqueness of the Laplace transform. □

Proposition 4.7. *Let $S_\alpha(t)$, $0 < \alpha \le 1$, $t \ge 0$, be a uniformly continuous solution operator satisfying $\|S_\alpha(t)\| \le Me^{\omega t}$, for some $M \ge 1$ and $\omega \ge 0$. Then*

(i) *There exists a unique bounded linear operator A such that*

$$S_\alpha(t) = E_\alpha(t^\alpha A), \quad t \ge 0.$$

(ii) *The operator A in (i) is the infinitesimal generator of solution operator $S_\alpha(t)$.*

(iii) *For every $t \ge 0$,*

$${}^C\mathcal{D}_t^\alpha S_\alpha(t) = AS_\alpha(t) = S_\alpha(t)A.$$

Proof. Fix $0 < \alpha \le 1$. From Theorem 4.5 we know that the infinitesimal generator of $S_\alpha(t)$ is a bounded linear operator A. Also, A is the infinitesimal generator of $E_\alpha(t^\alpha A)$ and therefore by Theorem 4.6, $S_\alpha(t) = E_\alpha(t^\alpha A)$. All others assertions of the proposition follow from (i). □

Integral representation stated in the next proposition will often be used in proving some auxiliary results as well as in proving our main result.

Proposition 4.8. *Let $0 < \alpha < 1$ and let $S_\alpha(t)$ be a solution operator generated by A. Then*

$$\int_0^t S_\alpha(t-\tau)\, {}^{RL}\mathcal{D}_\tau^{1-\alpha} f(\tau) d\tau = \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}((t-\tau)^\alpha A) f(\tau) d\tau. \tag{4.4}$$

Proof. Fix $0 < \alpha < 1$. Taking into account the relation 2.3 in Proposition 2.1 one gets

$$\begin{aligned}
\int_0^t S_\alpha(t-\tau)\, {}^{RL}\mathcal{D}_\tau^{1-\alpha} f(\tau) d\tau &= \int_0^t \sum_{n=0}^\infty \frac{1}{\Gamma(1+n\alpha)} (t-\tau)^{n\alpha} A^n\, {}^{RL}\mathcal{D}_\tau^{1-\alpha} f(\tau) d\tau \\
&= \sum_{n=0}^\infty \int_0^t \frac{1}{\Gamma(1+n\alpha)} (t-\tau)^{n\alpha} A^n\, {}^{RL}\mathcal{D}_\tau^{1-\alpha} f(\tau) d\tau \\
&= \sum_{n=0}^\infty J_t^{n\alpha+1} A^n\, {}^{RL}\mathcal{D}_t^{1-\alpha} f(t) = \sum_{n=0}^\infty J_t^{n\alpha+\alpha} A^n f(t) \\
&= \sum_{n=0}^\infty \frac{1}{\Gamma(n\alpha+\alpha)} \int_0^t (t-\tau)^{n\alpha+\alpha-1} A^n f(\tau) d\tau \\
&= \int_0^t (t-\tau)^{\alpha-1} \sum_{n=0}^\infty \frac{1}{\Gamma(n\alpha+\alpha)} (t-\tau)^{n\alpha} A^n f(\tau) d\tau \\
&= \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}((t-\tau)^\alpha A) f(\tau) d\tau.
\end{aligned}$$

□

Similarly, the first order derivative of the previously integral representation has the following form.

Proposition 4.9. *Let $0 < \alpha < 1$ and let $S_\alpha(t)$ be a solution operator generated by A. Then*

$$\begin{aligned}\frac{d}{dt}\int_0^t S_\alpha(t-\tau)\,{}^{RL}\mathcal{D}_\tau^{1-\alpha} f(\tau)d\tau &= \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}((t-\tau)^\alpha A)\partial_\tau f d\tau \\ &\quad + t^{\alpha-1}E_{\alpha,\alpha}(t^\alpha A)f(0).\end{aligned} \tag{4.5}$$

Proof. Fix $0 < \alpha < 1$. From the proof of Proposition 4.8 it follows that

$$\frac{d}{dt}\int_0^t S_\alpha(t-\tau)\,{}^{RL}\mathcal{D}_\tau^{1-\alpha} f(\tau)d\tau = \sum_{n=0}^\infty \frac{d}{dt} J_t^{n\alpha+\alpha} A^n f(t).$$

Further, since

$$\frac{d}{dt} J_t^\alpha f(t) = {}^{RL}\mathcal{D}_t^{1-\alpha} f(t) = {}^{C}\mathcal{D}_t^{1-\alpha} f(t) + \frac{f(0)t^{\alpha-1}}{\Gamma(\alpha)} = J_t^\alpha \frac{d}{dt} f(t) + \frac{f(0)t^{\alpha-1}}{\Gamma(\alpha)},$$

we have

$$\begin{aligned}\sum_{n=0}^\infty \frac{d}{dt} J_t^{n\alpha+\alpha} A^n f(t) &= \sum_{n=0}^\infty J_t^{n\alpha+\alpha} A^n \frac{d}{dt} f(t) + \sum_{n=0}^\infty \frac{J_t^{n\alpha} t^{\alpha-1} A^n f(0)}{\Gamma(\alpha)} \\ &= \sum_{n=0}^\infty J_t^{n\alpha+\alpha} A^n \frac{d}{dt} f(t) + \sum_{n=0}^\infty \frac{t^{n\alpha+\alpha-1}}{\Gamma(\alpha+n\alpha)} A^n f(0),\end{aligned}$$

and similarly to the proof of Proposition 4.8 one finally gets the relation (4.5). □

Motivated by Proposition 2.6 we give the fractional Duhamel principle in the case of solution operator.

Proposition 4.10. *The solution of the Cauchy problem* (1.1) *with Caputo fractional derivative is given by*

$$u(t) = S_\alpha(t)u_0 + \int_0^t S_\alpha(t-\tau)\,{}^{RL}\mathcal{D}_\tau^{1-\alpha} f(\cdot,\tau,u)d\tau, \tag{4.6}$$

where $S_\alpha(t)$ is a solution operator generated by A. The solution above is called mild solution to the problem (1.1).

Proof. Since ${}^{C}\mathcal{D}_t^\alpha S_\alpha(t) = AS_\alpha(t)$, for a continuous function its fractional integral J_t^α is a continuous function too and ${}^{C}\mathcal{D}_t^\alpha$ is a left inverse of fractional integral J_t^α for all $\alpha \geq 0$ and all continuous functions, it can be easily shown that $u(t)$ given by (4.6) satisfies the Cauchy problem (1.1). □

Remark 4.11. The solution of the Cauchy problem (1.1) can also be represented by Caputo fractional derivative, but in that case one must additionally suppose that $f(\cdot, 0, u_0) = 0$.

5. Generalized uniformly continuous solution operators

First, recall that every linear and bounded operator on Banach space E is a closed and densely defined operator in E. Therefore, all results in the previous section continue to be valid in the case of linear and bounded operators on Banach space.

Instead of the Cauchy problem (4.1) with closed and densely defined operator A, let us now consider fractional Cauchy problem given by

$$ {}^C\mathcal{D}_t^\alpha u(t) = \widetilde{A}u(t), \quad t > 0;\, u(0) = x, \tag{5.1} $$

where $\widetilde{A}$ is a generalized linear bounded operator.

Definition 5.1. Let $0 < \alpha < 1$. $S_\alpha \in \mathcal{SG}_\alpha([0,\infty) : \mathcal{L}(E))$ is called a Colombeau uniformly continuous solution operator for (5.1) if it has a representative $(S_\alpha)_\varepsilon$ which is a uniformly continuous solution operator for (5.1) and for every ε small enough.

Proposition 5.2. *Let $0 < \alpha < 1$ and let $(S_\alpha)_{1\varepsilon}$ and $(S_\alpha)_{2\varepsilon}$ be representatives of a generalized uniformly continuous solution operator S_α, with infinitesimal generators $\widetilde{A}_{1\varepsilon}$ and $\widetilde{A}_{2\varepsilon}$, respectively, for ε small enough. Then*

$$ \widetilde{A}_{1\varepsilon} - \widetilde{A}_{2\varepsilon} \in \mathcal{SN}(E). $$

Proof. Fix $0 < \alpha < 1$. Then we have

$$ \begin{aligned} \widetilde{A}_{1\varepsilon} - \widetilde{A}_{2\varepsilon} &= ({}^C\mathcal{D}_t^\alpha (S_\alpha)_{1\varepsilon})(t)\big|_{t=0} - ({}^C\mathcal{D}_t^\alpha (S_\alpha)_{2\varepsilon})(t)\big|_{t=0} \\ &= {}^C\mathcal{D}_t^\alpha ((S_\alpha)_{1\varepsilon} - (S_\alpha)_{2\varepsilon})(t)\big|_{t=0}. \end{aligned} $$

Since

$$ (S_\alpha)_{1\varepsilon} - (S_\alpha)_{2\varepsilon} \in \mathcal{SN}_\alpha([0,\infty) : \mathcal{L}(E)), $$

we have that, for every $a \in \mathbb{R}$, there exists $M > 0$ such that

$$ \|{}^C\mathcal{D}_t^\alpha ((S_\alpha)_{1\varepsilon} - (S_\alpha)_{2\varepsilon})(t)\big|_{t=0}\|_{\mathcal{L}(E)} \le M\varepsilon^a. $$

It implies that for every $a \in \mathbb{R}$ there exists $M > 0$ such that $\|\widetilde{A}_{1\varepsilon} - \widetilde{A}_{2\varepsilon}\| \le M\varepsilon^a$. Thus, $\widetilde{A}_{1\varepsilon} - \widetilde{A}_{2\varepsilon} \in \mathcal{SN}(E)$. ☐

Definition 5.3. $\widetilde{A} \in \mathcal{SG}(E)$ is called the infinitesimal generator of a Colombeau uniformly continuous solution operator $S_\alpha \in \mathcal{SG}_\alpha([0,\infty) : \mathcal{L}(E))$, $0 < \alpha < 1$, if $\widetilde{A}_\varepsilon$ is the infinitesimal generator of the representative $(S_\alpha)_\varepsilon$, for every ε small enough.

Proposition 5.4. *Let $0 < \alpha < 1$. Let $\widetilde{A}$ be the infinitesimal generator of a Colombeau uniformly continuous solution operator S_α, and $\widetilde{B}$ the infinitesimal generator of a Colombeau uniformly continuous solution operator T_α. If $\widetilde{A} = \widetilde{B}$, then $S_\alpha = T_\alpha$.*

Proof. Fix $0 < \alpha < 1$ and let $\widetilde{N}_\varepsilon = \widetilde{A}_\varepsilon - \widetilde{B}_\varepsilon \in \mathcal{SN}(E)$. Then from the property (iii) in Proposition 4.7 we obtain

$$ {}^C\mathcal{D}_t^\alpha ((S_\alpha)_\varepsilon - (T_\alpha)_\varepsilon)(t)x = \widetilde{A}_\varepsilon((S_\alpha)_\varepsilon - (T_\alpha)_\varepsilon)(t)x + \widetilde{N}_\varepsilon (T_\alpha)_\varepsilon(t)x. $$

By using fractional Duhamel principle (4.6) and since $(S_\alpha)_\varepsilon(0) = (T_\alpha)_\varepsilon(0) = I$, one gets

$$ ((S_\alpha)_\varepsilon - (T_\alpha)_\varepsilon)(t)x = \int_0^t (S_\alpha)_\varepsilon(t-\tau)\, {}^{RL}\mathcal{D}_\tau^{1-\alpha} \widetilde{N}_\varepsilon (T_\alpha)_\varepsilon(\tau) x\, d\tau. \tag{5.2} $$

Then, from the integral representation given in Proposition 4.8 we have

$$\begin{aligned}((S_\alpha)_\varepsilon-(T_\alpha)_\varepsilon)(t)x&=\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}((t-\tau)^\alpha\widetilde{A}_\varepsilon)\widetilde{N}_\varepsilon(T_\alpha)_\varepsilon(\tau)xd\tau\\&=\int_0^t(t-\tau)^{\alpha-1}\sum_{n=0}^\infty\frac{(t-\tau)^{n\alpha}\widetilde{A}_\varepsilon^n}{\Gamma(\alpha+n\alpha)}\widetilde{N}_\varepsilon(T_\alpha)_\varepsilon(\tau)xd\tau\\&=\int_0^t\sum_{n=0}^\infty\frac{(t-\tau)^{(n+1)\alpha-1}\widetilde{A}_\varepsilon^n}{\Gamma((n+1)\alpha)}\widetilde{N}_\varepsilon(T_\alpha)_\varepsilon(\tau)xd\tau\\&=\sum_{n=1}^\infty\int_0^t\frac{(t-\tau)^{n\alpha-1}\widetilde{A}_\varepsilon^{n-1}}{\Gamma(n\alpha)}\widetilde{N}_\varepsilon(T_\alpha)_\varepsilon(\tau)xd\tau.\end{aligned}$$

For $t\in[0,T)$, $T>0$, we obtain estimate

$$\begin{aligned}\|((S_\alpha)_\varepsilon-(T_\alpha)_\varepsilon)(t)\|&\le\sum_{n=1}^\infty\frac{1}{\Gamma(n\alpha)}\int_0^t(t-\tau)^{n\alpha-1}\|\widetilde{A}_\varepsilon^{n-1}\widetilde{N}_\varepsilon(T_\alpha)_\varepsilon(\tau)\|d\tau\\&\le\|\widetilde{N}_\varepsilon\|\sup_{t\in[0,T)}\|(T_\alpha)_\varepsilon(t)\|\sum_{n=1}^\infty\|\widetilde{A}_\varepsilon\|^{n-1}\frac{1}{\Gamma(n\alpha)}\frac{T^{n\alpha}}{n\alpha}\\&\le\|\widetilde{N}_\varepsilon\|\sup_{t\in[0,T)}\|(T_\alpha)_\varepsilon(t)\|\frac{T^\alpha}{\alpha}\sum_{n=0}^\infty\frac{T^{n\alpha}\|\widetilde{A}_\varepsilon\|^n}{\Gamma(\alpha+n\alpha)}\\&=\|\widetilde{N}_\varepsilon\|\sup_{t\in[0,T)}\|(T_\alpha)_\varepsilon(t)\|\frac{T^\alpha}{\alpha}E_{\alpha,\alpha}(T^\alpha\|\widetilde{A}_\varepsilon\|),\end{aligned}$$

and using the estimate (2.7) for $E_{\alpha,\alpha}$ we have

$$\begin{aligned}&\|((S_\alpha)_\varepsilon-(T_\alpha)_\varepsilon)(t)\|\\&\le\|\widetilde{N}_\varepsilon\|\sup_{t\in[0,T)}\|(T_\alpha)_\varepsilon(t)\|\frac{T^\alpha}{\alpha}C_\alpha(1+\|\widetilde{A}_\varepsilon\|^{(1-\alpha)/\alpha})(1+T^{1-\alpha})\exp(T\|\widetilde{A}_\varepsilon\|^{1/\alpha})\\&=\frac{C_\alpha}{\alpha}\|\widetilde{N}_\varepsilon\|\sup_{t\in[0,T)}\|(T_\alpha)_\varepsilon(t)\|(1+\|\widetilde{A}_\varepsilon\|^{(1-\alpha)/\alpha})(T+T^\alpha)\exp(T\|\widetilde{A}_\varepsilon\|^{1/\alpha}).\end{aligned}$$

Now, we consider the case $\gamma=\alpha$. For $t\in[0,T)$, $T>0$, one similarly gets

$$\begin{aligned}\|{}^C\mathcal{D}_t^\alpha((S_\alpha)_\varepsilon-(T_\alpha)_\varepsilon)(t)\|&\le\|\widetilde{N}_\varepsilon\|\sup_{t\in[0,T)}\|(T_\alpha)_\varepsilon(t)\|\sum_{n=0}^\infty\frac{\|\widetilde{A}_\varepsilon\|^n}{\Gamma(n\alpha)}\cdot\frac{T^{n\alpha}}{n\alpha}\\&=\|\widetilde{N}_\varepsilon\|\sup_{t\in[0,T)}\|(T_\alpha)_\varepsilon(t)\|\cdot E_\alpha(T^\alpha\|\widetilde{A}_\varepsilon\|).\end{aligned}$$

Differentiation of integral representation (5.2) with respect to t, using integral representation (4.5) one gets

$$\begin{aligned}&\frac{d}{dt}((S_\alpha)_\varepsilon-(T_\alpha)_\varepsilon)(t)x\\&=\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}((t-\tau)^\alpha\widetilde{A}_\varepsilon)\widetilde{N}_\varepsilon\frac{d}{d\tau}(T_\alpha)_\varepsilon(\tau)xd\tau+t^{\alpha-1}E_{\alpha,\alpha}(t^\alpha\widetilde{A}_\varepsilon)\widetilde{N}_\varepsilon x.\end{aligned}$$

Then, for every $T_1>0$ and $t\in[T_1,T)$, the estimate in norm is

$$\|\frac{d}{dt}((S_\alpha)_\varepsilon-(T_\alpha)_\varepsilon)(t)\|$$

$$
\begin{aligned}
&\le \lim_{\eta\to 0^+}\int_\eta^t (t-\tau)^{\alpha-1}\|E_{\alpha,\alpha}((t-\tau)^\alpha \widetilde{A}_\varepsilon)\widetilde{N}_\varepsilon \frac{d}{d\tau}(T_\alpha)_\varepsilon(\tau)\|d\tau\\
&\quad + t^{\alpha-1}E_{\alpha,\alpha}(t^\alpha\|\widetilde{A}_\varepsilon\|)\|\widetilde{N}_\varepsilon\|\\
&\le \lim_{\eta\to 0^+}\sup_{\tau\in[\eta,T)} E_{\alpha,\alpha}((T-\tau)^\alpha\|\widetilde{A}_\varepsilon\|)\|\widetilde{N}_\varepsilon\| \sup_{\tau\in[\eta,T)}\|\frac{d}{d\tau}(T_\alpha)_\varepsilon(\tau)\|\frac{(T-\eta)^\alpha}{\alpha}\\
&\quad + T_1^{\alpha-1}E_{\alpha,\alpha}(T^\alpha\|\widetilde{A}_\varepsilon\|)\|\widetilde{N}_\varepsilon\|.
\end{aligned}
$$

Finally, since $\widetilde{N}_\varepsilon \in \mathcal{SN}(E)$ it follows that for every $a\in\mathbb{R}$ there exists $M>0$ such that

$$
\sup_{t\in[0,T)}\|{}^C\mathcal{D}_t^\gamma((S_\alpha)_\varepsilon-(T_\alpha)_\varepsilon)(t)\|_{\mathcal{L}(E)}\le M\varepsilon^a,\ \gamma\in\{0,\alpha\},
$$

$$
\sup_{t\in(0,T)}\|\frac{d}{dt}((S_\alpha)_\varepsilon-(T_\alpha)_\varepsilon)(t)\|_{\mathcal{L}(E)}\le M\varepsilon^a,
$$

i.e. $(S_\alpha)_\varepsilon-(T_\alpha)_\varepsilon\in\mathcal{SN}_\alpha([0,\infty):\mathcal{L}(E))$. □

Definition 5.5. Let h_ε be a positive net satisfying $h_\varepsilon\le\varepsilon^{-1}$. It is said that $\widetilde{A}\in\mathcal{SG}(E)$ is of h_ε-type if it has a representative $\widetilde{A}_\varepsilon$ such that

$$
\|\widetilde{A}_\varepsilon\|_{\mathcal{L}(E)}=\mathcal{O}(h_\varepsilon),\quad \varepsilon\to 0.
$$

An element $G\in\mathcal{G}_\alpha([0,\infty):H^1(\mathbb{R}))$ is said to be of h_ε-type if it has a representative G_ε such that

$$
\|G_\varepsilon\|_{H^1}=\mathcal{O}(h_\varepsilon),\quad \varepsilon\to 0.
$$

The following proposition holds for generalized operators.

Proposition 5.6. *Let $0<\alpha<1$. Every $\widetilde{A}\in\mathcal{SG}(E)$ of h_ε-type, where $h_\varepsilon\le C(\log 1/\varepsilon)^\alpha$, is the infinitesimal generator of some generalized uniformly continuous solution operator $S_\alpha\in\mathcal{SG}_\alpha([0,\infty):\mathcal{L}(E))$.*

Proof. Fix $0<\alpha<1$. From Theorem 4.5 one knows that every linear and bounded operator $\widetilde{A}_\varepsilon$ is the infinitesimal generator of some uniformly continuous solution operator $(S_\alpha)_\varepsilon(t)$ defined by

$$
(S_\alpha)_\varepsilon(t)=E_\alpha(t^\alpha\widetilde{A}_\varepsilon)=\sum_{n=0}^\infty\frac{t^{n\alpha}\widetilde{A}_\varepsilon^n}{\Gamma(1+n\alpha)}.
$$

Let us show that $(S_\alpha)_\varepsilon\in\mathcal{S}E_M^\alpha([0,\infty):\mathcal{L}(E))$. From the inequality for Mittag-Leffler function it follows that there exists constant $M>0$ such that

$$
\|(S_\alpha)_\varepsilon(t)\|\le M\exp(t\|\widetilde{A}_\varepsilon\|^{1/\alpha}).
$$

Since $h_\varepsilon\le C(\log 1/\varepsilon)^\alpha$, we have

$$
\sup_{t\in[0,T)}\|(S_\alpha)_\varepsilon(t)\|\le M\varepsilon^{-TC^{1/\alpha}},
$$

for ε small enough. Also, since ${}^C\mathcal{D}_t^\alpha(S_\alpha)_\varepsilon(t)=\widetilde{A}_\varepsilon(S_\alpha)_\varepsilon(t)$, for every $t\ge 0$, we have for every ε small enough

$$
\|{}^C\mathcal{D}_t^\alpha(S_\alpha)_\varepsilon(t)\|\le\|\widetilde{A}_\varepsilon\|\|(S_\alpha)_\varepsilon(t)\|\le C(\log\frac{1}{\varepsilon})^\alpha M\varepsilon^{-TC^{1/\alpha}}\le CM\varepsilon^{-\alpha-TC^{1/\alpha}}.
$$

It remains to prove the moderate bound for $\|\frac{d}{dt}(S_\alpha)_\varepsilon(t)\|$. First, we have

$$\begin{aligned}\frac{d}{dt}(S_\alpha)_\varepsilon(t) &= \sum_{n=0}^{\infty}\frac{t^{(n+1)\alpha-1}}{\Gamma(\alpha+n\alpha)}\widetilde{A}_\varepsilon^{n+1}\\ &= t^{\alpha-1}\widetilde{A}_\varepsilon\sum_{n=0}^{\infty}\frac{t^{n\alpha}}{\Gamma(\alpha+n\alpha)}\widetilde{A}_\varepsilon^{n}\\ &= t^{\alpha-1}\widetilde{A}_\varepsilon E_{\alpha,\alpha}(t^\alpha\widetilde{A}_\varepsilon).\end{aligned}$$

Then, for every $T_1>0$ and $t\in[T_1,T)$, the estimate in norm is

$$\begin{aligned}\|\frac{d}{dt}(S_\alpha)_\varepsilon(t)\| &\le T_1^{\alpha-1}\|\widetilde{A}_\varepsilon\|E_{\alpha,\alpha}(t^\alpha\|\widetilde{A}_\varepsilon\|)\\ &\le T_1^{\alpha-1}\|\widetilde{A}_\varepsilon\|C_\alpha(1+\|\widetilde{A}_\varepsilon\|^{(1-\alpha)/\alpha})\cdot\exp(\|\widetilde{A}_\varepsilon\|^{1/\alpha}T)(1+T^{1-\alpha})\\ &\le T_1^{\alpha-1}C_\alpha(\|\widetilde{A}_\varepsilon\|+\|\widetilde{A}_\varepsilon\|^{1/\alpha})\cdot\exp(\|\widetilde{A}_\varepsilon\|^{1/\alpha}T)(1+T^{1-\alpha})\\ &\le T_1^{\alpha-1}C_\alpha((\log\frac{1}{\varepsilon})^\alpha+\log\frac{1}{\varepsilon})\cdot\exp(C^{1/\alpha}T\log\frac{1}{\varepsilon})(1+T^{1-\alpha})\\ &\le 2T_1^{\alpha-1}C_\alpha(1+T^{1-\alpha})\varepsilon^{-1-C^{1/\alpha}T}.\end{aligned}$$

Thus finally we have $(S_\alpha)_\varepsilon\in\mathcal{S}E_M^\alpha([0,\infty):\mathcal{L}(E))$. □

Note that a Colombeau uniformly continuous solution operator always possess an infinitesimal generator and it is unique. That follows from the fact that its representative is a classical uniformly continuous solution operator for which there exists a unique infinitesimal generator.

6. Existence and uniqueness result

In this section we specify the Banach space, i.e. we take $E=L^2(\mathbb{R})$. Instead of the Cauchy problem (4.1) with closed and densely defined operator A on $L^2(\mathbb{R})$ with domain $D(A)=H^1(\mathbb{R})$, we will consider fractional Cauchy problem given by

$${}^C\mathcal{D}_t^\alpha u(t)=\widetilde{A}u(t),\ t>0;\ u(0)=x,$$

where $\widetilde{A}$ is a generalized linear bounded operator L^2-associated with A, i.e., for every $u\in H^1(\mathbb{R})$, the following holds

$$\|(A-\widetilde{A}_\varepsilon)u\|_{L^2}\to 0,\quad \varepsilon\to 0.$$

Theorem 6.1. *Let $0<\alpha<1$. Suppose that $u_0\in\mathcal{G}_\alpha(H^1(\mathbb{R}))$ and let the function $f(x,t,u)$ be continuously differentiable with respect to t, globally Lipschitz with respect to x and u with bounded second order derivative with respect to u and $f(x,t,0)=0$. Also, suppose that $\partial_x f(x,t,u)$ and $\partial_t f(x,t,u)$ are globally Lipschitz function with respect to u. Let $g_1(x,t,u):=\partial_u f(x,t,u)$ and $g_2(x,t,u):=\partial_t f(x,t,u)$ satisfy the same conditions as $f(x,t,u)$.*

Let the operator $\widetilde{A}\in\mathcal{SG}(H^1(\mathbb{R}))$ be of h_ε-type, with $h_\varepsilon=o\big((\log(\log 1/\varepsilon)^\alpha)^\alpha\big)$, such that $\|\widetilde{A}_\varepsilon u_\varepsilon\|_{L^2}\le h_\varepsilon\|u_\varepsilon\|_{L^2}$, for $u_\varepsilon\in H^1(\mathbb{R})$.

Then for every $0<\alpha<1$ there exists a unique generalized solution $u\in\mathcal{G}_\alpha([0,\infty):H^1(\mathbb{R}))$ to the Cauchy problem

$${}^C\mathcal{D}_t^\alpha u(t)=\widetilde{A}u(t)+f(\cdot,t,u),\quad u(0)=u_0. \tag{6.1}$$

An equivalent integral equation for the solution (i.e. mild solution) is given by

$$u_\varepsilon(t) = (S_\alpha)_\varepsilon(t)u_{0\varepsilon} + \int_0^t (S_\alpha)_\varepsilon(t-\tau)^{RL}\mathcal{D}_\tau^{1-\alpha} f(\cdot,\tau,u_\varepsilon)d\tau, \tag{6.2}$$

where $S_\alpha \in \mathcal{SG}_\alpha([0,\infty) : \mathcal{L}(H^1(\mathbb{R})))$ is a Colombeau uniformly continuous solution operator generated by $\widetilde{A}$.

Remark 6.2. The existence of a solution for integral equation (6.2) can be proved using a Banach principle of a fixed point.

Proof of Theorem 6.1. Fix $0 < \alpha < 1$. Since the operator $\widetilde{A}$ is of h_ε-type, with $h_\varepsilon = o((\log\log 1/\varepsilon)^\alpha)$, it is obvious that the operator $\widetilde{A}$ is the infinitesimal generator of a Colombeau solution operator $S_\alpha \in \mathcal{SG}_\alpha([0,\infty) : \mathcal{L}(H^1(\mathbb{R})))$ given by $S_\alpha(t) = E_\alpha(t^\alpha \widetilde{A})$ (see Proposition 5.6). Also, from (4.6) we know that (6.2) represents a solution to (6.1).

Let us show that this solution is an element of $\mathcal{G}_\alpha([0,\infty) : H^1(\mathbb{R}))$. First, we show that the solution satisfies

$$\lim_{t\to 0^+} \|\frac{\frac{d}{dt}u_\varepsilon(t,\cdot)}{t^{\alpha-1}}\|_{H^1} = C < +\infty. \tag{6.3}$$

Indeed, after differentiation of (6.2) with respect to t, using the first order derivative of integral representation (4.5) one gets

$$\begin{aligned} &\frac{d}{dt}u_\varepsilon(t,\cdot) \\ &= \frac{d}{dt}(S_\alpha)_\varepsilon(t)u_{0\varepsilon} + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}((t-\tau)^\alpha \widetilde{A}_\varepsilon)\partial_\tau f(\cdot,\tau,u_\varepsilon(\tau))d\tau \\ &\quad + t^{\alpha-1}E_{\alpha,\alpha}(t^\alpha \widetilde{A}_\varepsilon) f(\cdot,0,u_{0\varepsilon}), \end{aligned} \tag{6.4}$$

and by to the notation $g_1(x,t,u) = \partial_u f(x,t,u)$ and $g_2(x,t,u) = \partial_t f(x,t,u)$, we have

$$\begin{aligned} \|\frac{d}{dt}u_\varepsilon(t,\cdot)\|_{L^2} &\le \|\frac{d}{dt}(S_\alpha)_\varepsilon(t)u_{0\varepsilon}\|_{L^2} \\ &\quad + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}((t-\tau)^\alpha\|\widetilde{A}_\varepsilon\|)\|\partial_\tau f(\cdot,\tau,u_\varepsilon(\tau))\|_{L^2}d\tau \\ &\quad + t^{\alpha-1}E_{\alpha,\alpha}(t^\alpha\|\widetilde{A}_\varepsilon\|)\|f(\cdot,0,u_{0\varepsilon})\|_{L^2} \\ &\le \|\frac{d}{dt}(S_\alpha)_\varepsilon(t)u_{0\varepsilon}\|_{L^2} \\ &\quad + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}((t-\tau)^\alpha\|\widetilde{A}_\varepsilon\|)\|g_1\|_{L^\infty}\|\partial_\tau u_\varepsilon(\tau)\|_{L^2}d\tau \\ &\quad + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}((t-\tau)^\alpha\|\widetilde{A}_\varepsilon\|)\|g_2\|_{L^\infty}\|u_\varepsilon(\tau)\|_{L^2}d\tau \\ &\quad + t^{\alpha-1}E_{\alpha,\alpha}(t^\alpha\|\widetilde{A}_\varepsilon\|)\|f(\cdot,0,u_{0\varepsilon})\|_{L^2}. \end{aligned} \tag{6.5}$$

After applying the Gronwall's inequality to (6.5) one gets

$$\lim_{t\to 0^+} \|\frac{\frac{d}{dt}u_\varepsilon(t,\cdot)}{t^{\alpha-1}}\|_{L^2} = C < +\infty.$$

Further, differentiation of (6.4) with respect to x, we have

$$
\begin{aligned}
&\|\partial_x \frac{d}{dt} u_\varepsilon(t,\cdot)\|_{L^2} \\
&\le \|\frac{d}{dt}(S_\alpha)_\varepsilon(t)\partial_x u_{0\varepsilon}\|_{L^2} \\
&\quad + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}((t-\tau)^\alpha \|\widetilde{A}_\varepsilon\|)\|\partial_x\partial_\tau f(\cdot,\tau,u_\varepsilon(\tau))\|_{L^2} d\tau \\
&\quad + t^{\alpha-1} E_{\alpha,\alpha}(t^\alpha\|\widetilde{A}_\varepsilon\|)\|\partial_x f(\cdot,0,u_{0\varepsilon})\|_{L^2} \\
&\le \|\frac{d}{dt}(S_\alpha)_\varepsilon(t)\partial_x u_{0\varepsilon}\|_{L^2} \\
&\quad + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}((t-\tau)^\alpha \|\widetilde{A}_\varepsilon\|)\|\partial_x g_1\|_{L^\infty}\|\partial_\tau u_\varepsilon(\tau)\|_{L^2} d\tau \qquad (6.6)\\
&\quad + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}((t-\tau)^\alpha \|\widetilde{A}_\varepsilon\|)\|g_1\|_{L^\infty}\|\partial_x\partial_\tau u_\varepsilon(\tau)\|_{L^2} d\tau \\
&\quad + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}((t-\tau)^\alpha \|\widetilde{A}_\varepsilon\|)\|\partial_x g_2\|_{L^\infty}\|u_\varepsilon(\tau)\|_{L^2} d\tau \\
&\quad + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}((t-\tau)^\alpha \|\widetilde{A}_\varepsilon\|)\|g_2\|_{L^\infty}\|\partial_x u_\varepsilon(\tau)\|_{L^2} d\tau \\
&\quad + t^{\alpha-1} E_{\alpha,\alpha}(t^\alpha\|\widetilde{A}_\varepsilon\|)[\|\partial_u f(\cdot,0,u_{0\varepsilon})\|_{L^\infty}\|\partial_x u_{0\varepsilon}\|_{L^2} \\
&\quad + \|\partial_x f(\cdot,0,u_{0\varepsilon})\|_{L^\infty}\|u_{0\varepsilon}\|_{L^2}].
\end{aligned}
$$

Again, after applying the Gronwall's inequality one gets

$$
\lim_{t\to 0^+} \|\frac{\partial_x \frac{d}{dt} u_\varepsilon(t,\cdot)}{t^{\alpha-1}}\|_{L^2} = C < +\infty,
$$

and finally we have that property (6.3) is satisfied.

Further, we prove that one has the moderate bound for $\|{}^C\mathcal{D}_t^\gamma u_\varepsilon(t,\cdot)\|_{H^1}$, $\gamma \in \{0,\alpha\}$, and $\|\frac{d}{dt}u_\varepsilon(t,\cdot)\|_{H^1}$. First, we prove the moderate bound for $\|{}^C\mathcal{D}_t^\gamma u_\varepsilon(t,\cdot)\|_{H^1}$, and consider the cases:

Case 1: $\gamma = 0$**.** From the representation (6.2) and Proposition 4.8 we obtain

$$
\|u_\varepsilon(t)\|_{L^2} \le \|(S_\alpha)_\varepsilon(t)u_{0\varepsilon}\|_{L^2} + \int_0^t (t-\tau)^{\alpha-1}\|E_{\alpha,\alpha}((t-\tau)^\alpha\widetilde{A}_\varepsilon)\|\,\|f(\cdot,\tau,u_\varepsilon)\|_{L^2} d\tau.
$$

Next, using the estimate for $E_{\alpha,\alpha}$ one gets

$$
\begin{aligned}
\|E_{\alpha,\alpha}(t^\alpha\widetilde{A}_\varepsilon)\| &\le \sum_{n=0}^\infty \frac{t^{n\alpha}\|\widetilde{A}_\varepsilon\|^n}{\Gamma(\alpha+n\alpha)} = E_{\alpha,\alpha}(t^\alpha\|\widetilde{A}_\varepsilon\|) \\
&\le C_\alpha(1+\|\widetilde{A}_\varepsilon\|^{(1-\alpha)/\alpha})(1+t^{1-\alpha})\exp(t\|\widetilde{A}_\varepsilon\|^{1/\alpha}).
\end{aligned}
$$

Denote

$$
\widetilde{M_T} := \sup_{t\in[0,T)} \|E_{\alpha,\alpha}(t^\alpha\widetilde{A}_\varepsilon)\|. \qquad (6.7)
$$

Note that for $\alpha = 1$ it follows $\widetilde{M_T} := \sup_{t\in[0,T)}\|S(t)\|$, where $S(t)$ is a generalized uniformly continuous semigroup of operators generated by the operator $\widetilde{A}$ (see [14]).

Next,

$$\begin{aligned}\widetilde{M}_T &\le C_\alpha(1+o((\log\log 1/\varepsilon)^{1-\alpha}))(1+T^{1-\alpha})\exp(T\cdot o(\log\log 1/\varepsilon))\\ &= \mathcal{O}(\log 1/\varepsilon),\end{aligned} \tag{6.8}$$

by the well known properties of Landau's symbol o. From

$$\begin{aligned}\|u_\varepsilon(t)\|_{L^2} &\le \|(S_\alpha)_\varepsilon(t)u_{0\varepsilon}\|_{L^2} + \widetilde{M}_T\int_0^t (t-\tau)^{\alpha-1}\|f(\cdot,\tau,u_\varepsilon)\|_{L^2}d\tau\\ &\le \|(S_\alpha)_\varepsilon(t)u_{0\varepsilon}\|_{L^2} + C\widetilde{M}_T\int_0^t (t-\tau)^{\alpha-1}\|u_\varepsilon(\tau)\|_{L^2}d\tau,\end{aligned}$$

using Gronwall's inequality we obtain the moderate bound for $\|u_\varepsilon(t)\|_{L^2}$.

After differentiation of (6.2) with respect to x, using similar integral representation as the one in Proposition 4.8 we have

$$\partial_x u_\varepsilon(t) = (S_\alpha)_\varepsilon(t)\partial_x u_{0\varepsilon} + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}((t-\tau)^\alpha \widetilde{A}_\varepsilon)\partial_x f(\cdot,\tau,u_\varepsilon)d\tau$$

and

$$\begin{aligned}&\|\partial_x u_\varepsilon(t)\|_{L^2}\\ &\le \|(S_\alpha)_\varepsilon(t)\partial_x u_{0\varepsilon}\|_{L^2} + \int_0^t (t-\tau)^{\alpha-1}\|E_{\alpha,\alpha}((t-\tau)^\alpha\widetilde{A}_\varepsilon)\|\|\partial_x(f(\cdot,\tau,u_\varepsilon))\|_{L^2}d\tau\\ &\le \|(S_\alpha)_\varepsilon(t)\partial_x u_{0\varepsilon}\|_{L^2} + \widetilde{M}_T\int_0^t (t-\tau)^{\alpha-1}\|\partial_u f\|_{L^\infty}\|\partial_x u_\varepsilon(\tau)\|_{L^2}d\tau\\ &\quad + \widetilde{M}_T\int_0^t (t-\tau)^{\alpha-1}\|\partial_x f\|_{L^\infty}\|u_\varepsilon(\tau)\|_{L^2}d\tau.\end{aligned}$$

Since f is Lipschitz with respect to u and x the moderate bound for $\|\partial_x u_\varepsilon(t)\|_{L^2}$ again follows from the Gronwall's inequality.

Case 2: $\gamma = \alpha$. From (6.1) we have

$$\|{}^C\mathcal{D}_t^\alpha u_\varepsilon(t)\|_{L^2} \le \|\widetilde{A}_\varepsilon u_\varepsilon(t)\|_{L^2} + \|f(\cdot,t,u_\varepsilon)\|_{L^2}.$$

Since f is globally Lipschitz with respect to u and $f(x,t,0)=0$, it follows the moderate bound for $\|{}^C\mathcal{D}_t^\alpha u_\varepsilon(t)\|_{L^2}$.

Differentiation of (6.1) with respect to x we have

$$\begin{aligned}\|\partial_x^C\mathcal{D}_t^\alpha u_\varepsilon(t)\|_{L^2} &\le \|\partial_x(\widetilde{A}_\varepsilon u_\varepsilon(t))\|_{L^2} + \|\partial_x(f(\cdot,t,u_\varepsilon))\|_{L^2}\\ &\le C(\log 1/\varepsilon)^\alpha\|u_\varepsilon(t)\|_{H^1} + \|\partial_u f\|_{L^\infty}\|\partial_x u_\varepsilon(t)\|_{L^2}\\ &\quad + \|\partial_x f\|_{L^\infty}\|u_\varepsilon(t)\|_{L^2},\end{aligned}$$

and the moderate bound for $\|\partial_x^C\mathcal{D}_t^\alpha u_\varepsilon(t)\|_{L^2}$ immediately follows.

The moderate bound for $\|\frac{d}{dt}u_\varepsilon(t,\cdot)\|_{H^1}$ follows after applying the Gronwall's inequality to inequalities (6.5) and (6.6).

To prove that this solution is unique in Colombeau space $\mathcal{G}_\alpha([0,\infty):H^1(\mathbb{R}))$, suppose that there exist two solutions u and v to (6.1) and set $\omega_\varepsilon = u_\varepsilon - v_\varepsilon$. This difference satisfies

$${}^C\mathcal{D}_t^\alpha\omega_\varepsilon(t) = \widetilde{A}_\varepsilon\omega_\varepsilon(t) + f(\cdot,t,u_\varepsilon) - f(\cdot,t,v_\varepsilon) + \widetilde{N}_\varepsilon(t), \quad \omega_\varepsilon(0) = \omega_{0\varepsilon}, \tag{6.9}$$

where $\widetilde{N}_\varepsilon(t) \in \mathcal{N}_\alpha([0,\infty) : H^1(\mathbb{R}))$ and $\omega_{0\varepsilon} \in \mathcal{N}_\alpha(H^1(\mathbb{R}))$. Then

$$\begin{aligned} \omega_\varepsilon(t) = (S_\alpha)_\varepsilon(t)\omega_{0\varepsilon} + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}((t-\tau)^\alpha \widetilde{A}_\varepsilon)(f(\cdot,\tau,u_\varepsilon) \\ - f(\cdot,\tau,v_\varepsilon))d\tau + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}((t-\tau)^\alpha \widetilde{A}_\varepsilon)\widetilde{N}_\varepsilon(\tau)d\tau, \end{aligned} \tag{6.10}$$

and

$$\begin{aligned} \|\omega_\varepsilon(t)\|_{L^2} \le \|(S_\alpha)_\varepsilon(t)\omega_{0\varepsilon}\|_{L^2} + \int_0^t (t-\tau)^{\alpha-1} \|E_{\alpha,\alpha}((t-\tau)^\alpha \widetilde{A}_\varepsilon)\| \cdot \|f(\cdot,\tau,u_\varepsilon) \\ - f(\cdot,\tau,v_\varepsilon)\|_{L^2} d\tau + \int_0^t (t-\tau)^{\alpha-1} \|E_{\alpha,\alpha}((t-\tau)^\alpha \widetilde{A}_\varepsilon)\| \cdot \|\widetilde{N}_\varepsilon(\tau)\|_{L^2} d\tau. \end{aligned}$$

Since $\|E_{\alpha,\alpha}((t-\tau)^\alpha \widetilde{A}_\varepsilon)\| \le \widetilde{M}_T$, $0 \le t \le T$, $0 \le \tau \le t$, where $\widetilde{M}_T$ is estimated by (6.8) and since f is a Lipschitz function with respect to u, we obtain the $\mathcal{N}$-bound for $\|\omega_\varepsilon(t)\|_{L^2}$.

Equation (6.9) implies

$$\|{}^C\mathcal{D}_t^\alpha \omega_\varepsilon(t)\|_{L^2} \le \|\widetilde{A}_\varepsilon \omega_\varepsilon(t)\|_{L^2} + \|f(\cdot,t,u_\varepsilon) - f(\cdot,t,v_\varepsilon)\|_{L^2} + \|\widetilde{N}_\varepsilon(t)\|_{L^2},$$

and the $\mathcal{N}$-bound for $\|{}^C\mathcal{D}_t^\alpha \omega_\varepsilon(t)\|_{L^2}$ immediately follows.

Differentiation of (6.10) with respect to x we have

$$\begin{aligned} &\|\partial_x \omega_\varepsilon(t)\|_{L^2} \\ &\le \|(S_\alpha)_\varepsilon(t)\partial_x \omega_{0\varepsilon}\|_{L^2} \\ &\quad + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}((t-\tau)^\alpha \|\widetilde{A}_\varepsilon\|) \|\partial_u f(\cdot,\tau,u_\varepsilon)\partial_x u_\varepsilon - \partial_u f(\cdot,\tau,v_\varepsilon)\partial_x v_\varepsilon\|_{L^2} d\tau \\ &\quad + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}((t-\tau)^\alpha \|\widetilde{A}_\varepsilon\|) \|\partial_x f(\cdot,\tau,u_\varepsilon)u_\varepsilon - \partial_x f(\cdot,\tau,v_\varepsilon)v_\varepsilon\|_{L^2} d\tau \\ &\quad + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}((t-\tau)^\alpha \|\widetilde{A}_\varepsilon\|) \|\partial_x \widetilde{N}_\varepsilon(\tau)\|_{L^2} d\tau. \end{aligned}$$

However, f has bounded second order derivative with respect to u, and similarly to [14] we have

$$\begin{aligned} &\|\partial_u f(\cdot,\tau,u_\varepsilon)\partial_x u_\varepsilon - \partial_u f(\cdot,\tau,v_\varepsilon)\partial_x v_\varepsilon\|_{L^2} \\ &\le C_1 \|\partial_x u_\varepsilon(\tau)\|_{L^2} \|\omega_\varepsilon(\tau)\|_{H^1} + C_2 \|\partial_x \omega_\varepsilon(\tau)\|_{L^2}. \end{aligned}$$

Also, since $\partial_x f$ is Lipschitz with respect to u we have

$$\begin{aligned} &\|\partial_x f(\cdot,\tau,u_\varepsilon)u_\varepsilon - \partial_x f(\cdot,\tau,v_\varepsilon)v_\varepsilon\|_{L^2} \\ &\le \|\partial_x f(\cdot,\tau,u_\varepsilon)\|_{L^\infty} \|\omega_\varepsilon(\tau)\|_{L^2} + \|v_\varepsilon(\tau)\|_{H^1} \|\partial_u^2 f(\cdot,\tau,\widetilde{y})\|_{L^\infty} \|\omega_\varepsilon(\tau)\|_{L^2}, \end{aligned}$$

for some function $\widetilde{y} \in H^1(\mathbb{R})$, and the $\mathcal{N}$-bound for $\|\partial_x \omega_\varepsilon(t)\|_{L^2}$ follows from the Gronwall's inequality.

Differentiation of (6.9) with respect to x yields

$$\|\partial_x^C \mathcal{D}_t^\alpha \omega_\varepsilon(t)\|_{L^2} \le \|\partial_x(\widetilde{A}_\varepsilon \omega_\varepsilon(t))\|_{L^2} + \|\partial_x(f(\cdot,t,u_\varepsilon) - f(\cdot,t,v_\varepsilon))\|_{L^2} + \|\partial_x \widetilde{N}_\varepsilon(t)\|_{L^2},$$

and the $\mathcal{N}$-bound for $\|\partial_x^C \mathcal{D}_t^\alpha \omega_\varepsilon(t)\|_{L^2}$ immediately follows.

The $\mathcal{N}$-bound for $\|\frac{d}{dt}\omega_\varepsilon(t)\|_{H^1}$ can be obtained in a similar manner: first by differentiating equation (6.10) with respect to t, then differentiating this new equation with respect to x, and, at the end, by applying the Gronwall's inequality.

Finally, it follows that $\omega_\varepsilon := u_\varepsilon - v_\varepsilon \in \mathcal{N}_\alpha([0,\infty) : H^1(\mathbb{R}))$, i.e. the solution is unique. $\square$

Remark 6.3. If $\widetilde{A} \in \mathcal{SG}(H^2(\mathbb{R}))$ is an operator of h_ε-type with

$$h_\varepsilon = o\Big((\log(\log 1/\varepsilon)^\alpha)^\alpha\Big),$$

similarly one can prove that solution to (6.1) is also represented by (6.2) and this unique solution belongs to $\mathcal{G}_\alpha([0,\infty) : H^2(\mathbb{R}))$.

Definition 6.4. The solution u of problem (6.1) introduced in Theorem 6.1 is called generalized solution of the equation

$${}^C\mathcal{D}_t^\alpha u(t) = \widetilde{A}u(t) + f(\cdot, t, u)$$

with generalized operators.

7. Comparison of solutions to the original and approximate problems

In this section we prove that, under certain additional conditions, the solutions of problem (4.1) and corresponding approximate problem (5.1) are L^2-associated.

Theorem 7.1. *Let $0 < \alpha < 1$. Assume that there exists the solution, $u_\varepsilon \in H^2(\mathbb{R})$, of the equation*

$${}^C\mathcal{D}_t^\alpha u_\varepsilon(x,t) = Au_\varepsilon(x,t) + f(x,t,u_\varepsilon(x,t)), \quad t>0,\ x\in\mathbb{R}, \quad u_\varepsilon(0) = u_{0\varepsilon}, \tag{7.1}$$

where A is a closed linear operator densely defined in the Banach space $L^2(\mathbb{R})$ with domain $D(A) = H^1(\mathbb{R})$ and property $A : H^2(\mathbb{R}) \to H^1(\mathbb{R})$. Let v_ε be a solution of the corresponding approximate equation with the same initial data:

$${}^C\mathcal{D}_t^\alpha v_\varepsilon(x,t) = \widetilde{A}_\varepsilon v_\varepsilon(x,t) + f(x,t,v_\varepsilon(x,t)), \quad t>0,\ x\in\mathbb{R},\ v_\varepsilon(0) = u_{0\varepsilon}, \tag{7.2}$$

where f and $\widetilde{A} \in \mathcal{S}G(H^1(\mathbb{R}))$ are given as in Theorem 6.1. Additionally, let the generalized operator $\widetilde{A}$ satisfies:

- (i) $\|\widetilde{A}_\varepsilon u_\varepsilon\|_{L^2} \le C\|u_\varepsilon\|_{H^1}$, *for $u_\varepsilon \in H^2(\mathbb{R})$, where C does not depend on ε.*
- (ii) $\|(A - \widetilde{A}_\varepsilon)u_\varepsilon\|_{H^1} \to 0$, *for $u_\varepsilon \in H^2(\mathbb{R})$, when $\varepsilon \to 0$.*

Then the solutions u_ε and v_ε are L^2-associated, i.e., for every $T > 0$,

$$\sup_{t\in[0,T)} \|u_\varepsilon(t) - v_\varepsilon(t)\|_{L^2} \to 0, \quad \text{as } \varepsilon \to 0.$$

Remark 7.2. The generalized operator $\widetilde{A}$ satisfying properties (i) and (ii) can be obtained, for instance, by regularization of space fractional or integer order derivatives appearing in the operator A. For details we refer to [14].

Proof of Theorem 7.1. Fix $0 < \alpha < 1$. Since u_ε and v_ε satisfy the equations (7.1) and (7.2), respectively, one gets

$$\begin{aligned} {}^C\mathcal{D}_t^\alpha (u_\varepsilon(x,t) - v_\varepsilon(x,t)) &= \widetilde{A}_\varepsilon(u_\varepsilon(x,t) - v_\varepsilon(x,t)) + (A - \widetilde{A}_\varepsilon)u_\varepsilon(x,t) \\ &\quad + f(x,t,u_\varepsilon) - f(x,t,v_\varepsilon). \end{aligned} \tag{7.3}$$

Put $\omega_\varepsilon = u_\varepsilon - v_\varepsilon$. Then (7.3) becomes

$${}^C\mathcal{D}_t^\alpha \omega_\varepsilon(t) = \widetilde{A}_\varepsilon \omega_\varepsilon(t) + f(\cdot,t,u_\varepsilon) - f(\cdot,t,v_\varepsilon) + N_\varepsilon(t), \quad \omega_\varepsilon(0) = 0,$$

where $N_\varepsilon(t) = (A - \widetilde{A}_\varepsilon)u_\varepsilon(\cdot, t)$. Then ω_ε satisfies

$$\begin{aligned}\omega_\varepsilon(t) &= \int_0^t (S_\alpha)_\varepsilon(t-\tau)^{RL}\mathcal{D}_\tau^{1-\alpha}(f(\cdot,\tau,u_\varepsilon) - f(\cdot,\tau,v_\varepsilon))d\tau \\ &\quad + \int_0^t (S_\alpha)_\varepsilon(t-\tau)^{RL}\mathcal{D}_\tau^{1-\alpha}N_\varepsilon(\tau)d\tau,\end{aligned}$$

and from integral representation (4.4) it follows that

$$\begin{aligned}\omega_\varepsilon(t) &= \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}((t-\tau)^\alpha \widetilde{A}_\varepsilon)(f(\cdot,\tau,u_\varepsilon) - f(\cdot,\tau,v_\varepsilon))d\tau \\ &\quad + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}((t-\tau)^\alpha \widetilde{A}_\varepsilon)N_\varepsilon(\tau)d\tau.\end{aligned}$$

The estimation in the norm gives

$$\begin{aligned}\|\omega_\varepsilon(t)\|_{L^2} &\le \int_0^t (t-\tau)^{\alpha-1}\|E_{\alpha,\alpha}((t-\tau)^\alpha \widetilde{A}_\varepsilon)(f(\cdot,\tau,u_\varepsilon) - f(\cdot,\tau,v_\varepsilon))\|_{L^2}d\tau \\ &\quad + \int_0^t (t-\tau)^{\alpha-1}\|E_{\alpha,\alpha}((t-\tau)^\alpha \widetilde{A}_\varepsilon)N_\varepsilon(\tau)\|_{L^2}d\tau.\end{aligned}$$

By assumption (i) we have $\|\widetilde{A}_\varepsilon u_\varepsilon\|_{L^2} \le C\|u_\varepsilon\|_{H^1}$, where C does not depend on ε. Therefore

$$\|E_{\alpha,\alpha}(t^\alpha \widetilde{A}_\varepsilon)u_\varepsilon\|_{L^2} \le E_{\alpha,\alpha}(t^\alpha C)\|u_\varepsilon\|_{H^1},$$

for $u_\varepsilon \in H^2(\mathbb{R})$. Further, from the assumption (ii) it follows

$$\|E_{\alpha,\alpha}((t-\tau)^\alpha \widetilde{A}_\varepsilon)N_\varepsilon(\tau)\|_{L^2} \to 0,$$

as $\varepsilon \to 0$. Using that $\|\partial_u f\|_{L^\infty} \le C_1 < \infty$ and the estimate

$$\|f(\cdot,s,u_\varepsilon) - f(\cdot,s,v_\varepsilon)\|_{L^2} \le \|\partial_u f\|_{L^\infty} \cdot \|u_\varepsilon(s) - v_\varepsilon(s)\|_{L^2} \le C_1\|\omega_\varepsilon(s)\|_{L^2},$$

Gronwall's inequality gives $\sup_{t\in[0,T)}\|\omega_\varepsilon(t)\|_{L^2} \to 0$, as $\varepsilon \to 0$. □

Remark 7.3. The similar result can be obtained in the case when A is a closed linear operator densely defined in the Banach space $L^2(\mathbb{R})$ with domain $D(A) = H^2(\mathbb{R})$ and property $A : H^4(\mathbb{R}) \to H^2(\mathbb{R})$, assuming that there exists the solution $u_\varepsilon \in H^4(\mathbb{R})$.

8. Applications to fractional differential equations with space variable coefficients

In this section we give the explicit examples and illustrate how one can obtain the approximate operator $\tilde{A}$ for a given (integer or fractional) differential operator A. In these examples, the corresponding generalized operators will be in the form of regularized operators. The regularization is necessary in order to transform unbounded differential operators into bounded operators. In all examples that we list below, one can prove that the operators A and $\widetilde{A}$ satisfy similar properties (i) and (ii) from Theorem 7.1 (for details we refer [14]).

8.1. **Time fractional reaction-diffusion equation.** Let $0 < \alpha < 1$ and let $f(x,t,u(x,t))$ describes the outer force in the Cauchy problem for equation with space variable coefficients, i.e.

$$ {}^C\mathcal{D}_t^\alpha u(x,t) = \lambda(x)\partial_x^2 u(x,t) + f(x,t,u(x,t), \quad t>0,\ x\in\mathbb{R}, $$

where the function f satisfy conditions from the Theorem 6.1 and $\lambda(x)$ is a such that the operator $A = \lambda(x)\partial_x^2$ satisfies the conditions from Remark 7.3 (for example, one can choose $\lambda \in L^\infty(\mathbb{R})$). The equation of this type is very important in the theory of fractional Brownian motion and anomalous transport of premises [27]. Also, this equation is used in population biology to model the spread of invasive species. In that case, $u(x,t)$ is the population density at location $x \in \mathbb{R}$ and time $t > 0$. The first term on the right-hand side is the diffusion term ($\lambda(x)$ is a diffusion coefficient) and it models migration, while the second term $f(x,t,u(x,t))$ is the reaction term that models population growth.

Instead of the previous problem let us consider the corresponding approximate problem

$$ {}^C\mathcal{D}_t^\alpha u(x,t) = \widetilde{A}u(x,t) + f(x,t,u(x,t)), \quad t>0,\ x\in\mathbb{R}, $$

where the operator $\widetilde{A} \in \mathcal{SG}(H^2(\mathbb{R}))$ is represented by the nets of operators

$$ \widetilde{A}_\varepsilon : H^2(\mathbb{R}) \to H^2(\mathbb{R}), $$
$$ \widetilde{A}_\varepsilon u_\varepsilon = \lambda_\varepsilon(x)(\partial_x^2 u_\varepsilon * \phi_{h_\varepsilon}), $$

such that $\lambda_\varepsilon \in H^2(\mathbb{R})$, $\|\lambda_\varepsilon\|_{H^2(\mathbb{R})} = \mathcal{O}\Big((\log(\log 1/\varepsilon)^\alpha)^{\alpha/2}\Big)$, $\phi_{h_\varepsilon}(x) = h_\varepsilon\phi(xh_\varepsilon)$, where $h_\varepsilon = o\Big((\log(\log 1/\varepsilon)^\alpha)^{\alpha/5}\Big)$, $\phi \in C_0^\infty(\mathbb{R})$, $\phi(x) \geq 0$ and $\int \phi(x)dx = 1$.

Then, the mild solution is given by (6.2) and the solution belongs to Colombeau space $\mathcal{G}_\alpha([0,\infty) : H^2(\mathbb{R}))$.

8.2. **Time-space fractional reaction-diffusion equation.** Instead of the Cauchy problem for the time-space fractional equation with variable coefficients and with f satisfying conditions from the Theorem 6.1, let us consider the corresponding approximate problem, i.e.

$$ {}^C\mathcal{D}_t^\alpha u(t) = \widetilde{A}_\beta u(t) + f(\cdot,t,u), $$

where $0 < \alpha < 1$, $1 < \beta < 2$, the operator $\widetilde{A}_\beta \in \mathcal{SG}(H^2(\mathbb{R}))$ is represented by the nets of operators

$$ (\widetilde{A}_\beta)_\varepsilon : H^2(\mathbb{R}) \to H^2(\mathbb{R}), $$
$$ (\widetilde{A}_\beta)_\varepsilon u_\varepsilon = \lambda_\varepsilon(x)(\mathcal{D}_+^\beta u_\varepsilon * \phi_{h_\varepsilon}), $$

where $\mathcal{D}_+^\beta$ is the left Liouville fractional derivative of order β on the whole axis $\mathbb{R}$ given by

$$ (\mathcal{D}_+^\beta u)(x) = \frac{1}{\Gamma(2-\beta)}\big(\frac{d}{dx}\big)^2 \int_{-\infty}^x \frac{u(\xi)}{(x-\xi)^{\beta-1}} d\xi, $$

$\lambda_\varepsilon \in H^2(\mathbb{R})$ and $\phi_{h_\varepsilon}(x)$ satisfies the same properties as in the case of time fractional diffusion equation.

Then, the mild solution is given by (6.2) and the solution belongs to Colombeau space $\mathcal{G}_\alpha([0,\infty) : H^2(\mathbb{R}))$. The same result holds if instead of left βth Liouville fractional derivative in the fractional operator $\widetilde{A}_\beta$, $1 < \beta < 2$, one uses right βth Liouville fractional derivative or Riesz βth fractional derivative.

8.3. **Time-space fractional reaction-advection-diffusion equation.** Let $0 < \alpha < 1$, $0 < \beta \le 1$, $1 < \gamma \le 2$ and consider fractional equation

$$ {}^C\mathcal{D}_t^\alpha u(t) = -a(x)\mathcal{D}_+^\beta u(t) + b(x)\mathcal{D}_+^\gamma u(t) + f(\cdot, t, u), $$

where $a(x)$ and $b(x)$ are such that corresponding differential operator A again satisfies the conditions from Remark 7.3.

Such equation has a physical meaning, since it is an appropriate model for many interesting phenomena. For example, it models the transport of a chemical or biological tracer carried by water through a medium that is uniform, porous and saturated. In that case, u is a solute concentration, $a(x)$ and $b(x)$ represent fluid velocity and the dispersion, respectively, while f is a given contaminant source which is common in hydrogeological phenomena.

Again we consider the corresponding approximate Cauchy problem for the time-space fractional reaction-advection-diffusion equation with variable coefficients and with f satisfying conditions from the Theorem 6.1, i.e.

$$ {}^C\mathcal{D}_t^\alpha u(t) = \widetilde{A}_{\beta,\gamma} u(t) + f(\cdot, t, u), $$

where $0 < \alpha < 1$, $0 < \beta \le 1$, $1 < \gamma \le 2$ and the operator $\widetilde{A}_{\beta,\gamma} \in \mathcal{SG}(H^2(\mathbb{R}))$ is represented by the nets of operators

$$ (\widetilde{A}_{\beta,\gamma})_\varepsilon : H^2(\mathbb{R}) \to H^2(\mathbb{R}), $$
$$ (\widetilde{A}_{\beta,\gamma})_\varepsilon u_\varepsilon = -a_\varepsilon(x)(\mathcal{D}_+^\beta u_\varepsilon * \phi_{h_\varepsilon}) + b_\varepsilon(x)(\mathcal{D}_+^\gamma u_\varepsilon * \phi_{h_\varepsilon}), $$

assuming that functions a_ε and b_ε satisfy similar conditions as λ_ε in time fractional diffusion equation.

The mild solution is given by (6.2), the solution belongs to Colombeau space $\mathcal{G}_\alpha([0,\infty) : H^2(\mathbb{R}))$ and one can uses right Liouville fractional derivative or Riesz fractional derivative instead of the left Liouville fractional derivative.

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