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ON THE ϕ_0-STABILITY OF IMPULSIVE DYNAMIC SYSTEM ON TIME SCALES

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Abstract. This paper concerns impulsive dynamic system on time scales. We obtain sufficient conditions of the ϕ_0-stability for such systems by employing cone-valued Lyapunov functions.

1. Introduction

Dynamic systems with impulses have been subject of numerous investigations, and are used as mathematical models of various real processes and phenomena in physics, biology, control theory and so on, which during their evolutionary processes experience an abrupt change of state at certain moments of time. Furthermore, the theory of such systems is much richer than the corresponding theory of differential systems without impulses. The past ten years have seen a significant development in the theory of impulsive differential equations. For the basic theory and recent developments, see the monograph [4] and references cited therein.

Recently, the stability theory of dynamic equations on time scales is emerging as an important area of investigation since it demonstrates the interplay of the two different theories, namely, the theories of continuous and discrete dynamic systems [5]. However, the corresponding theory of such equations is still at an initial stage of its development, especially the impulsive dynamic system on time scales. For the various of stability problem, it is well known that the method of cone-valued Lyapunov functions is beneficial in applications and circumvents the limitations of the useful and well-known method of both the scalar and vector Lyapunov functions. Lakshmikantham and Leela [6] introduced the concept of cone-valued Lyapunov functions. Since then, Akpan [3], Akinyele [1], Akinyele and Adeyeye [2], Soliman [8] investigated the stability and ϕ_0-stability of the differential systems, impulsive control systems, hybrid systems and perturbed impulsive systems by employing the method of cone-valued Lyapunov functions. However, there are few results for the stability of various of impulsive dynamic system on time scales [7]. In this paper, utilizing the framework of the theory of dynamic systems on time scale, we establish

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new comparison results in terms of cone-valued Lyapunov functions and investigate impulsive dynamic system on time scales relative to ϕ_0-stability.

2. Preliminaries

Let $\mathbb{T}$ be a time scale (any subset of $\mathbb{R}$ with order and topological structure defined in a canonical way) with $t_0 \geq 0$ as minimal element and no maximal element. The basic concepts on time scales can be seen in [5].

Definition 2.1. The mapping $g : \mathbb{T} \to X$, where X is a Banach space, is called rd-continuous if at each right-dense $t \in \mathbb{T}$, it is continuous and at each left-dense t, the left-sided limit $g(t^-)$ exists.

Definition 2.2. For each $t \in \mathbb{T}$, let N be a neighborhood of t, Then, we define the generalized derivative (or Dini derivative), $D^+u^\Delta(t)$, to mean that, given $\varepsilon > 0$, there exists a right neighborhood $N_\varepsilon \subset N$ of t such that

$$\frac{u(\sigma(t)) - u(t)}{\mu(t,s)} < D^+u^\Delta(t) + \varepsilon, \quad \text{for } s \in N_\varepsilon, \quad s > t, \quad \text{where } \mu(t,s) \equiv \sigma(t) - s.$$

When t is the right-scattered and u is continuous at t, we have, as in the case of the derivative

$$D^+u^\Delta(t) = \frac{u(\sigma(t)) - u(t)}{\mu^*(t)},$$

where $\mu^*(t) = \sigma(t) - t$.

Definition 2.3. A proper subset K of $\mathbb{R}^n$ is called a cone if

(i) $\lambda K \subseteq K,\ \lambda \geq 0$;
(ii) $K + K \subseteq K$
(iii) $K = \overline{K}$
(iv) $K^0 \neq \emptyset$
(v) $K \cap (-K) = \{0\}$,

where $\overline{K}$ and K^0 denote the closure and interior of K, respectively, and ∂K denotes the boundary of K. The order relation on $\mathbb{R}^n$ induced by the cone K is defined as follows: $x \leq_K y$ iff $y - x \in K$ and $x <_{K^0} y$ if and only if $y - x \in K^0$.

Definition 2.4. The set $K^* = \{\phi \in \mathbb{R}^n : (\phi, x) \geq 0, \text{ for all } x \in K\}$ is said to be adjoint cone if it satisfies the properties (i)-(v).

$$x \in K^0 \quad \text{if} \quad (\phi, x) > 0,$$

and

$$x \in \partial k \quad \text{if} \quad (\phi, x) = 0, \quad \text{for some } \phi \in K_0^*,\ K_0 = K - \{0\}.$$

Definition 2.5. A function $g : D \to \mathbb{R}^n$, $D \in \mathbb{R}^n$ is said to be quasimonotone relative to K if $x, y \in D$ and $y - x \in \partial K$ implies that there exists $\phi_0 \in K_0^*$ such that

$$(\phi_0, y - x) = 0 \quad \text{and} \quad (\phi_0, g(y) - g(x)) \geq 0.$$

3. Comparison results

Consider the impulsive dynamic system

$$\begin{gathered} x^{\Delta} = f(t,x), \quad t \in \mathbb{T},\ t \neq t_k, \\ x(t^+) = x(t) + I_k(x), \quad t = t_k,\ k \in \mathbb{N} \\ x(t_0^+) = x_0, \quad t_0 \geq 0, \end{gathered} \tag{3.1}$$

under the following assumptions:

(i) $0 < t_1 < t_2 < \dots < t_k < \dots$, and $\lim_{k\to\infty} t_k = \infty$

(ii) $f \in C_{rd}[\mathbb{T}\times\mathbb{R}^n, \mathbb{R}^n]$ is rd-continuous in $(t_{k-1}, t_k]\times\mathbb{R}^n$ and for each $x \in \mathbb{R}^n$, $k \in \mathbb{N}$, $\lim_{(t,y)\to(t_k^+,x)} f(t,y) = f(t_k^+, x)$

(iii) $I_k \in C_{rd}[\mathbb{R}^n, \mathbb{R}^n]$.

We will assume for the remainder of the paper that, for each $k \in \mathbb{N}$, the points of impulsive t_k are right-dense. To define the solution of (3.1), we shall consider the space

$$\begin{aligned} PC = \Big\{ & x : \mathbb{T} \to \mathbb{R}^n : x_k \in C_{rd}[[t_{k-1}, t_k], \mathbb{R}^n],\ k \in \mathbb{N} \text{ and there exist} \\ & x(t_k^-) \text{ and } x(t_k^+),\ k \in \mathbb{N} \text{ with } x(t_k^-) = x(t_k) \Big\}, \end{aligned}$$

which is a Banach space with the norm

$$\|x\|_{PC} = \sup\{|x_k|,\ k = 0,1,2,\dots\},$$

where x_k is the restriction of x to $[t_{k-1}, t_k]$, $k \in \mathbb{N}$.

Definition 3.1. A function $x \in PC \cap \bigcup_{k=1}^{\infty} C_{rd}[(t_{k-1}, t_k), \mathbb{R}^n]$ is said to be a solution of (3.1), if it satisfies

$$x^{\Delta} = f(t,x) \quad \text{everywhere on } \mathbb{T}\setminus\{t_k\},\ k \in \mathbb{N},$$

and for each $k \in \mathbb{N}$ the function x satisfies the conditions $x(t_k^+) - x(t_k^-) = I_k(x_k)$ and the initial condition $x(t_0^+) = x_0$.

For $\rho > 0$, define $S(\rho) = \{x \in \mathbb{R}^n : \|x\| < \rho\}$. Let $V \in C_{rd}[\mathbb{T}\times S(\rho), K]$. Then V is said to belong to the class V_0 if V is locally Lipschitzian in x and rd-continuous in $(t_{k-1}, t_k]\times S(\rho)$, and for each $x \in S(\rho)$, $k \in \mathbb{N}$, $\lim_{(t,y)\to(t_k^+,x)} V(t,y) = V(t_k^+, x)$ exists.

Following Definition 2.2, we say that $V \in C_{rd}[\mathbb{T}\times S(\rho), K]$, $D^+V^{\Delta}(t, x(t))$ if for each $\varepsilon > 0$, there exists a right neighborhood $N_\varepsilon \subset N$ of t such that

$$\frac{1}{\mu(t,s)}[V(\sigma(t), x(\sigma(t))) - V(s, x(\sigma(t))) - \mu(t,s) f(t, x(t))] < D^+V^{\Delta}(t, x(t)) + \varepsilon$$

for each $s \in N_\varepsilon$, $s > t$. As before, if t is right-scattered and $V(t, x(t))$ is continuous at t, this reduces to

$$D^+V^{\Delta}(t, x(t)) = \frac{V(\sigma(t), x(\sigma(t))) - V(t, x(t))}{\mu^*}.$$

Lemma 3.2 ([4]). *Let $m \in C_{rd}[\mathbb{T}, \mathbb{R}^n]$ be a mapping that is differentiable for each $t \in \mathbb{T}$ and that satisfies*

$$m^{\Delta}(t,x) \leq g(t, m(t)),\ t \in \mathbb{T},$$

where $g \in C_{rd}[\mathbb{T}, \mathbb{R}^n]$ *and* $g(t,u)\mu^*(t)$ *be nondecreasing in* u *for each* $t \in \mathbb{T}$. *Then* $m(t_0) \le u_0$ *implies*

$$m(t) \le r(t), \quad t \in \mathbb{T},$$

where $r(t)$ *is the maximal solution of* $u^\Delta = g(t,u)$, $u(t_0) = u_0 \ge 0$ *existing on* $\mathbb{T}$.

Consider the comparison system

$$\begin{gathered} u^\Delta = g(t,u), \quad t \in \mathbb{T},\ t \neq t_k, \\ u(t_k^+) = J_k(u(t_k)), \quad k \in \mathbb{N} \\ u(t_0^+) = u_0 \ge 0. \end{gathered} \tag{3.2}$$

Assume that

(i) $g \in C_{rd}[\mathbb{T} \times K, \mathbb{R}^n]$, g is rd-continuous in $(t_{k-1}, t_k] \times K$, K is a cone in $\mathbb{R}^n$, and for each $u \in K$, $\lim_{(t,v)\to(t_k^+,u)} g(t,v) = g(t_k^+, u)$, $g(t,u)\mu^*(t) + u$ is quasimonotone nondecreasing in u relative to K for each (t,u)

(ii) $V \in C_{rd}[\mathbb{T} \times S(\rho), K]$, V is locally Lipschitzian in x relative to K and
$$D^+V^\Delta(t,x) \le_K g(t, V(t,x)), \quad t \neq t_k$$

(iii) $J_k \in C_{rd}[K,K]$, and J_k is quasimonotone nondecreasing relative to K such that
$$V(t, x + I_k(x)) \le_K J_k(V(t,x)), \quad t = t_k.$$

Lemma 3.3. *Assume (i)–(iii) above, and let* $r(t,t_0,u_0)$ *be the maximal solution of* (3.2) *existing on* $\mathbb{T}$. *Then for any solution* $x(t) = x(t,t_0,x_0)$ *of* (3.1) *which exists on* $\mathbb{T}$, *we have*

$$V(t,x(t)) \le_K r(t,t_0,u_0) \quad \textit{provided that} \quad V(t_0^+, x_0) \le_K u_0.$$

Proof. Let $x(t) = x(t,t_0,x_0)$ be the solution of (3.1) existing for $t \ge t_0$, $t \in \mathbb{T}$, and set $m(t) = V(t,x(t))$. Then by assumption (ii), it is easy to derive

$$D^+m^\Delta(t) \le_K g(t, m(t)).$$

For $t \in [t_0, t_1]$, $m(t_0) = V(t_0, x_0) = u_0$. Then we get by Lemma 3.2

$$m(t) \le_K r_0(t,t_0,u_0),$$

where $r_0(t,t_0,u_0)$ is the maximal solution of (3.2) with $r_0(t_0^+, t_0, u_0) = u_0$. Since $J_1(u)$ is quasimonotone nondecreasing relative to K, and by (iii),

$$m(t_1) \le_K J_1(m(t_1)) \le_K r_0(t_1,t_0,u_0) = u_1^+,$$

where $u_1^+ \le J_1(r_0(t_1,t_0,u_0))$. By (3), (3) and Lemma 3.2,

$$m(t) \le_K r_1(t,t_1,u_1^+), \quad t \in (t_1, t_2],$$

where $r_1(t,t_1,u_1^+)$ is the maximal solution of (3.2) with $r_1(t_1^+, t_1, u_1^+) = u_1^+$. This procedure can be repeated successively to arrive at

$$m(t) \le_K r_k(t,t_k,u_k^+), \quad t \in (t_k, t_{k+1}],$$

where $r_k(t,t_k,u_k^+) = u_k^+$ for each $k = 0,1,2,\dots$. It is clear that $m(t) \le_K u(t)$, $t \ge t_0$ and $m(t) \le_K r(t,t_0,u_0)$, $t \ge t_0$. Therefore,

$$V(t,x(t)) \le_K r(t,t_0,u_0), \quad t \ge t_0,\ t \in \mathbb{T}.$$

The proof is complete. $\square$

4. Main results

Definition 4.1. The trivial solution $u = 0$ of (3.2) is ϕ_0-equistable if for each $\varepsilon > 0$ and $t_0 \in \mathbb{T}$, there exists a positive function $\delta = \delta(t_0, \varepsilon)$ that is rd-continuous in t_0 for each ε such that for $\phi_0 \in K_0^*$

$$(\phi_0, u_0) < \delta \quad \text{implies} \quad (\phi_0, r(t)) < \varepsilon, \quad t \geq t_0,\ t \in \mathbb{T}.$$

In the above definition, and for the rest of this paper, $r(t)$ denotes the maximal solution of (3.2) relative to the cone $K \subseteq \mathbb{R}^n$. Other ϕ_0-stability concepts can be similarly defined.

Definition 4.2. The trivial solution $u = 0$ of (3.2) is uniformly ϕ_0-stable if δ in Definition 4.1 is independent of t_0.

Definition 4.3. The trivial solution $u = 0$ of (3.2) is ϕ_0-equi-asymptotically stable if it is ϕ_0-equistable, and for each $\varepsilon > 0$, $t_0 \in \mathbb{T}$, there exist $\delta_0 = \delta_0(t_0) > 0$ and $T = T(t_0, \varepsilon)$ such that

$$(\phi_0, u_0) < \delta_0 \quad \text{implies} \quad (\phi_0, r(t)) < \varepsilon,\ t \geq t_0 + T(t_0, \varepsilon),\ t \in \mathbb{T}.$$

Definition 4.4. The trivial solution $u = 0$ of (3.2) is uniformly ϕ_0-asymptotically stable if δ and T in Definition 4.3 is independent of t_0.

Definition 4.5. A function $a(r)$ is said to belong to the class κ if $a \in C_{rd}[\mathbb{R}_+, \mathbb{R}_+]$, $a(0) = 0$, and $a(r)$ is strictly monotone increasing function in r.

Next we discuss stability criteria for the trivial solution of (3.1) under the assumptions

(iv) $f(t, 0) = 0$, $g(t, 0) = 0$, and for some $\phi_0 \in K_o^*$, $(t, x) \in \mathbb{T} \times S(\rho)$,

$$b(\|x\|) \leq (\phi_0, V(t, x)) \leq a(\|x\|), \quad a, b \in \kappa.$$

(v) There exists a ρ_o such that $x \in S(\rho_0)$ implies that $x + I_k(x) \in S(\rho)$ for all K.

Theorem 4.6. *In addition to the hypothesis of Lemma 3.3, we assume that (iv), (v) are satisfied. Then the ϕ_0-stability properties of the trivial solution of the comparison system* (3.2) *imply the corresponding stability properties of the trivial solution of* (3.1).

Proof. Let $\varepsilon \in (0, \rho_0)$ and $t_0 \in \mathbb{T}$ be given. Suppose that the trivial solution of (3.2) is ϕ_0-stable. Then given $b(\varepsilon) > 0$, and $t_0 \in \mathbb{T}$, there exists a $\delta^* = \delta^*(t_0, \varepsilon) > 0$ such that

$$(\phi_0, u_0) < \delta^* \quad \text{implies} \quad (\phi_0, u(t, t_0, u_0)) < b(\varepsilon), \quad t \in \mathbb{T}$$

where $u(t) = u(t, t_0, u_0)$ is any solution of (3.2). Choose $\delta = \delta(t_0, \varepsilon) > 0$ such that

$$a(\delta) < \delta^*.$$

We claim that if $\|x_0\| < \delta$, then $\|x(t)\| < \varepsilon$, $t \in \mathbb{T}$, where $x(t) = x(t, t_0, x_0)$ is any solution of (3.1). If this is not true, there would exist a $t^* \in \mathbb{T}$, $t^* > t_0$ such that $t_k < t^* \leq t_{k+1}$ for some k, and a solution $x(t) = x(t, t_0, x_0)$ of (3.1) satisfying

$$\varepsilon \leq \|x(t^*)\| \quad \text{and} \quad \|x(t)\| < \varepsilon \quad \text{for } t_0 \leq t \leq t_k. \tag{4.1}$$

Since $0 < \varepsilon < \rho_0$, it follows from assumption (v) that $\|x_k^+\| = \|x_k + I_k(x_k)\| < \rho$. Hence, we can find a $t^{**} \in (t_k, t^*]$ such that $\varepsilon \leq \|x(t^{**})\| < \rho$. Setting $m(t) =$

$V(t, x(t))$ for $t_0 \le t \le t^{**}$ and using condition (ii), we get by Lemma 3.3, the estimate

$$V(t, x(t)) \le_K r(t, t_0, u_0), \quad t_0 \le t \le t^{**}.$$

Now the relations (4), (4.1), (4) and the assumption (iv) yield

$$b(\varepsilon) \le b(\|x(t^{**})\|) \le (\phi_0, V(t^{**}, x(t^{**}))) \le (\phi_0, r(t^{**}, t_0, x_0)) < b(\varepsilon),$$

since $(\phi_0, u_0) = (\phi_0, V(t_0, x_0)) \le a(\|x_0\|) < a(\delta) < \delta^*$ by (4). This contradiction proves the claim. Thus the trivial solution of (3.1) is stable.

Now, suppose that the trivial solution of (3.2) is ϕ_0-asymptotically stable. Let $\varepsilon \in (0, \rho_0)$ be given. Then there exist $\delta^* = \delta^*(t_0, \varepsilon) > 0$ and $T > 0$ such that

$$(\phi_0, u_0) < \delta^* \quad \text{implies} \quad (\phi_0, u(t, t_0, u_0)) < b(\varepsilon), \quad t \in \mathbb{T},\ t \ge t_0 + T.$$

We assert that for any solution $x(t) = x(t, t_0, x_0)$ of (3.1) with $\|x_0\| < \delta$, $\|x(t)\| < \varepsilon$, $t \ge t_0 + T$, where δ is the same as (4). If this is not true, then there would exist a solution $x(t)$ of (3.1) and a divergent sequence $\{t_n\}$, $t_n \ge t_0 + T$, with the property $\|x(t_n)\| \ge \varepsilon$. Since we have

$$V(t, x(t)) \le_K r(t, t_0, u_0), \quad t \ge t_0,$$

it follows that

$$b(\varepsilon) \le b(\|x(t_n)\|) \le (\phi_0, V(t_n, x(t_n))) \le (\phi_0, r(t_n, t_0, x_0)) < b(\varepsilon), \quad t \ge t_0 + T,$$

which is a contradiction. Thus the trivial solution of system (3.1) is ϕ_0-asymptotically stable. The proof of other stability properties is similar. □

Corollary 4.7. *The function $g(t, u) = 0$ is admissible in Theorem 4.6 to yield uniform stability of the trivial solution of* (3.1).

Corollary 4.8. *The choice of the function $g(t, u) = -c(t)$, $c \in \kappa$ in Theorem 4.6 implies uniform asymptotic stability of the trivial solution of* (3.1).

Proof. Uniform stability of (3.1) follows from Corollary 4.2. Then, set $\varepsilon = \rho$, for some $\rho > 0$ and designate by $\delta_0 = \delta_0(\rho)$ so that we have

$$x_0 < \delta \quad \text{implies} \quad \|x(t)\| < \rho, \quad t \ge t_0,\ t \in \mathbb{T}.$$

Setting

$$L(t, x(t)) = V(t, x(t)) + \int_{t_0}^{t} c[V(s, x(s))]\Delta s,$$

we see that

$$D^+ L^\Delta(t, x(t)) = D^+ V^\Delta(t, x) + c[V(t, x)] \le g(t, V(t, x)) + c[V(t, x)] = 0.$$

Then

$$L(t, x(t)) \le L(t_0, x_0) = V(t_0, x_0), \quad t \ge t_0. \tag{4.2}$$

Choose $T = T(\varepsilon) > 0$, for any given $\varepsilon > 0$, to satisfy $T > a(\delta_0)/c[b(\delta)]$, where $\delta = \delta(\varepsilon)$. Because of the trivial solution of (3.1) is uniformly ϕ_0-stable, it is sufficient to show that there exists a $t^* \in [t_0, t_0 + T]$ such that $\|x(t^*)\| < \delta$. If this is not true, we should have $\|x(t)\| \geq \delta$, $t \in [t_0, t_0+T]$ and hence $(\phi_0, V(t, x(t))) \geq b(\delta)$, $t \in [t_0, t_0+T]$. It then follows from (4.2)

$$\begin{aligned}
a(\delta_0) &< Tc[b(\delta)] \\
&\leq (\phi_0, V(t_0+T, x(t_0+T))) + \int_{t_0}^{t_0+T} (\phi_0, c[V(s, x(s))])\Delta s \\
&\leq (\phi_0, V(t_0, x_0)) \\
&\leq a(\|x_0\|) \leq a(\delta_0),
\end{aligned}$$

which is a contradiction. The proof is complete. □

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