Electronic Journal of Differential Equations, Vol. **2001**(2001), No. 49, pp. 1–4.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

Note on the uniqueness of a global positive solution to the second Painlevé equation *

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Abstract

The purpose of this note is to study the uniqueness of solutions to $u'' - u^3 + (t-c)u = 0$, for $t \in (0,+\infty)$ with Neumann condition at 0. Assuming a certain conditon at infinity, Helfer and Weissler [6] have found a unique solution. We show that, without any assumptions at infinity, this problem has exactly one global positive solution. Moreover, the solution behaves like $\sqrt{t}$ as t approaches infinity.

1 Introduction

The existence and uniqueness of solution to the Painlevé equation

$$u'' = u^3 - (t-c)u, \tag{1.1}$$

posed in the semi-infinite interval $(0,+\infty)$, with a Neumann condition at 0

$$u'(0) = 0, \tag{1.2}$$

and having a prescribed behavior at $+\infty$

$$u(t) \approx \sqrt{t}, \tag{1.3}$$

has recently been considered by Helffer and Weissler [6]. Equation (1.1) appears in the study of the superheating field attached to a semi-infinite superconductor [2, 6]. When $c = 0$, Equation (1.1) has a connection with the Korteweg-de Vries equation; see [1, 7]. The following theorem presents the family of solutions to (1.1)–(1.3), in terms of c, obatined by Helfer and Weissler [6].

Theorem 1.1 *For each $c \in \mathbb{R}$, there exists a unique solution, u_c, to* (1.1)–(1.3). *This solution is positive, strictly increasing and at infinity it satisfies*

$$u_c(t) = \sqrt{t} + O(t^{-5/2}), \tag{1.4}$$

and

$$u_c'(t) = 2^{-1/2}(t)^{-1/2} + O(t^{-2}). \tag{1.5}$$

* *Mathematics Subject Classifications:* 34B15, 35B05, 82D55.
Key words: Second Painlevé equation, Neumann condition, global existence.

Submitted February 08, 2001. Published July 9, 2001.

The proof is similar to the one used by Hastings and McLeod [5] for constructing the unique strictly positive solution, defined on $\mathbb{R}$, to

$$u'' + tu - u^3 = 0, \tag{1.6}$$

such that $\lim_{t\to-\infty} u(t) = 0$ and $u(t) \approx \sqrt{t}$ at $+\infty$.

The main objective of the present note is to prove the uniqueness of a global positive solution to (1.1)–(1.2) without any conditions at $+\infty$.

2 Main Result

As in [6], $u(.,\alpha)$ denotes the unique maximal solution $u \in C^2((0,T(\alpha)),\mathbb{R})$, to (1.1)–(1.2) satisfying $u(0,\alpha) = \alpha$. To prove Theorem 1.1 Helffer and Weissler showed the existence of a unique $\alpha = \alpha(c)$ such that $u(.,\alpha(c))$ is global, positive and the quantity $u(t,\alpha(c)) - \sqrt{t}$ tends to 0 as t approaches $+\infty$. The parameter $\alpha(c)$ satisfies

$$0 < \alpha(c)(\alpha(c)^2 + c) \le 1.$$

The idea of the proof is to demonstrate that

$$\mathcal{N} = (0,\alpha(c)), \quad \text{and} \quad \mathcal{P} = (\alpha(c),+\infty),$$

where

$$\begin{aligned}
\mathcal{P} &= \big\{\alpha > 0; u(.,\alpha) > 0 \text{ on } [0,T(\alpha)), \text{ and} \\
&\quad u(.,\alpha) > h(t) \text{ on } (t_0,T(\alpha)) \text{ for some } t_0 \in (0,T(\alpha))\big\}, \\
\mathcal{N} &= \big\{\alpha > 0; \text{ there exists } 0 < t_0 < T(\alpha), u(.,\alpha) > 0 \text{ on } [0,t_0) \\
&\quad \text{and } u(t_0,\alpha) = 0\big\},
\end{aligned}$$

with

$$h(t) = \sqrt{(t-c)_+}.$$

Our main result is the following.

Theorem 2.1 *For every $c \in \mathbb{R}$ there exists a unique global positive solution, u_g, to*

$$\begin{gathered}
u'' = u^3 - (t-c)u, \qquad t \in (0,+\infty), \\
u'(0) = 0\,.
\end{gathered} \tag{2.1}$$

Moreover

$$\lim_{t\to\infty}(u_g(t) - \sqrt{t}) = 0,$$

and then

$$u_g \equiv u(.,\alpha(c)).$$

This theorem is an immediate consequence of Theorem 1.1 and of the following propositon.

Proposition 2.1 *For every $\alpha \in \mathcal{P}$, the maximal interval of definition $[0,T(\alpha))$ satisfies $T(\alpha) < +\infty$.*

Proof. Let $\alpha \in \mathcal{P}$. Assume on the contrary that $u(.,\alpha) =: u$ is global. Since $u > h$ for t large, u goes to infinity with $t, u' > 0, u'' > 0$ for t large and the limit $\lim_{t\to+\infty} u'(t)$ exists in $(0, +\infty]$. Next fix $\varepsilon \in (0,1)$. Because $\lim_{t\to\infty} \frac{\sqrt{\varepsilon}u'(t)}{h'(t)} = +\infty$ we deduce that

$$\lim_{t\to\infty} \frac{\sqrt{\varepsilon}u(t)}{h(t)} = +\infty,$$

thanks to the l'Hôpital rule. Therefore,

$$u'' = u(u^2 - h^2) \geq (1-\varepsilon)u^3,$$

for t large, and then u is not global. This is a contradiction that completes the Proof. □

Remark 2.1 Now it is clear that the unique global positive solution to (1.1)–(1.2) is the one required by Chapman; this confirms the previous condition at infinity. By similar argument, we can prove that any global positive solution to (1.1) satisfies (1.3) at infinity.

Remark 2.2 In the same spirit, we can show that the problem

$$(|u'|^{p-2}u')' = u|u|^{p-2}\left(|u|^q - |h|^{q-1}h\right),$$
$$u'(0) = 0, \quad p > 1, \quad q > 0,$$

possesses a unique positive global solution, under some restrictions on h [3]. Moreover, this solution behaves like h at infinity.

Remark 2.3 A similar classification is obtained in [4] for the problem

$$|y'|^{p-2}y' = y^q - \beta y.$$

This equation is satisfied by similarity solutions to

$$u_t = (|u_x|^{p-2}u_x)_x - (u^q)_x, \quad q = 2(p-1).$$

Acknowledgments. The author is grateful for a partial support from the Department of Mathematics of the Faculty of Sciences and Technique, FST Marrakech Morroco, during his visit there. This work was also partially supported by DRI (UPJV) Amiens, France.

References

[1] M. J. Ablowitz and H. Segur, *Exact linearization of a Painlevé transcendent,* Phys. Rev. Lett., 38, (1977), pp. 1103–1106.

[2] S. J. Chapman, *Superheating field of type II superconductors,* SIAM J. Appl. Math.. 55, (1995), pp. 1233–1258.

[3] A. Gmira and M. Guedda, *Classification of solutions to a class of nonlinear differential equations,* International J. Diff. Equat. Appl., Vol 1., No 2, (2000), pp. 223–238,

[4] A. Gmira, M. Guedda et L. Veron, *Source-type solution for the one-dimensional diffusion-convection equation,* NoDEA, 7, No. 2, (2000), pp. 127–142.

[5] S. P. Hastings and J. B. McLeod, *A boundary value problem associated with the second Painlevé transcendent and the Korteweg-de Vries equation,* Arch. Rational Mech. Anal., 73, No. 1, (1980), pp. 31–51.

[6] B. Helffer and F. B. Weissler, *On a family of solutions of the second Painlevé equation related to superconductivity,* European J. Appl. Math., Vol. 9, No. 3, (1998), pp. 223–243.

[7] R. M. Miura, *The Korteweg-de Vries equation; a survey of results,* SIAM Rev., 18, (1976), pp. 412–459.

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