ON THE POWER DOMINATION PROBLEM IN GRAPHS

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On the Power Domination Problem in Graphs

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Abstract

A crucial task for electric power companies consists of the continuous monitoring of their power network. This monitoring can be efficiently accomplished by placing phase measurement units (PMUs) at selected network locations. However, due to the high cost of the PMUs, their number must be minimized [1]. Finding the minimum number of PMUs needed to monitor a given power network, as well as to determine the locations where the PMUs should be placed, give rise to the power domination problem in graph theory [8].

The power dominating problem is NP-complete, that is, there is no efficient way of finding a minimal power dominating set for a graph. However, closed formulas for the power domination number of certain families of graphs, such as rectangular grids [5] have been found.

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Table of Contents

List of Figures v

Chapter 1. Introduction 1

Chapter 2. Definitions and Notation 3

Chapter 3. Power Domination 11

3.1 Structural Properties Related to γ_P 13

3.2 Known Bounds and Equalities for γ_P 17

Chapter 4. The Generalized Petersen Graph $\gamma_P(P(m, k))$ 20

Chapter 5. The Corona and the Power Domination Number 23

Chapter 6. An Extension of Power Domination 25

6.1 $k-$Power Domination 25

6.2 Extensions of Results from Power Domination 27

Chapter 7. Conclusion 30

List of Figures

2.1 G, an example of a graph 3

2.2 Interesting Families of Graphs 5

(a) P_4 5

(b) C_4 5

(c) bipartite graph 5

(d) $K_{3,3}$ 5

2.3 Three isomorphic K_4 graphs 6

(a) K_4 6

(b) K_4 6

(c) K_4 6

2.4 A subgraph and an induced subgraph 6

(a) Graph G 6

(b) Subgraph of G 6

(c) Induced Subgraph of G 6

2.5 A graph and its complement 7

(a) Graph G 7

(b) $\overline{G}$ 7

2.6 The Coronas of K_3 and K_1 7

(a) K_3 7

(b) K_1 7

(c) $K_3 \odot K_1$ 7

(d) $K_1 \odot K_3$ 7

2.7 A Subdivision of K_4 8

(a) K_4 8

(b) Subdivision of K_4 8

2.8 Graphs Homeomorphic to $K_{3,3}$ 9

(a) $K_{3,3}$ 9

(b) Homeomorphic to $K_{3,3}$ 9

(c) Homeomorphic to $K_{3,3}$ 9

2.9 The Cartesian Product of P_6 and P_4 9

(a) P_6 9

(b) P_4 9

(c) $P_6 \times P_4$ 9

2.10 An Example of Domination in Graphs 10

(a) Petersen Graph 10

(b) Dominating Set 10

(c) The Petersen Graph is dominated 10

3.1 Propagation Rules Illustrated 15

(a) Graph G 15

(b) PMU is placed on a vertex 15

(c) Neighborhood is observed 15

(d) Propagation Occurs 15

(e) Propagation Occurs . . . 15

(f) End of Propagation . . . 15

3.2 Homeomorphic Graphs With Different Power Domination Numbers . . . 16

(a) $\gamma_P(T) = 2$ . . . 16

(b) $\gamma_P(T') = 3$ . . . 16

3.3 The 8×10 grid, $P_8 \times P_{10}$ . . . 18

4.1 The Generalized Petersen Graph $P(8, 3)$ . . . 20

5.1 A Specific Case of the Class of Graphs Defined in Corollary 5.0.4 . . . 24

(a) $\gamma_P = 2$ . . . 24

(b) $\gamma_P = 3$ . . . 24

6.1 Propagation Rules Illustrated for $3-$Power Domination . . . 27

(a) Graph G . . . 27

(b) PMU is placed on a vertex . . . 27

(c) Neighborhood is observed . . . 27

(d) Propagation Occurs . . . 27

(e) Propagation Occurs . . . 27

(f) End of Propagation . . . 27

Chapter 1

Introduction

A crucial task for electric power companies consists of the continuous monitoring of their power network. This monitoring can be efficiently accomplished by placing phase measurement units (PMUs) at selected network locations. However, due to the high cost of the PMUs, the number of PMUs used to monitor the network must be minimized [1]. Finding the minimum number of PMUs needed to monitor a given power network, as well as determining the locations where the PMUs should be placed, gives rise to the power domination problem in graph theory [8].

The power domination problem is a relatively new problem in graph theory. The power dominating problem is NP-complete. However, closed formulas for the power domination number of certain families of graphs, such as rectangular grids [5] have been found.

We begin by first defining a graph. This is followed by the definitions that are needed in order to understand the power domination problem in graphs. In essence, the first chapter will be a brief excursion through the fundamentals of graph theory.

In the following chapter, the power domination problem will formally be introduced in terms of graph theory. The definition of the power domination number of a graph will be presented. This will be followed by giving some known results on the power domination problem. Structural results are given followed by some known bounds and equalities on the power domination number for various families of graph.

Next, a discussion on the Generalized Petersen Graphs will be given. The definition and a characterization of the isomorphism classes of the Generalized Petersen Graphs are recalled. Conjectures about the power domination number for the Generalized Petersen Graphs are also presented.

The corona, a binary operation between graphs, is recalled. Some identities that relate the power domination number to the (regular) domination number are shown, followed by a corollary that, with further investigation, may lead to some insight on the power domination problem.

An extension of power domination, $k-$power domination, in graphs is then defined. Some of the results from power domination are also extended into $k-$power domination.

Lastly, some open problems that one might want to explore are presented. These problems are either extensions of known results or are problems that may lead to further insight on the power domination problem.

Chapter 2

Definitions and Notation

Denote the cardinality of a set S as $|S|$. A graph is a pair $G = (V, E)$ where V is the vertex set of G and E is a set of doubleton subsets of V called the edge set of G. The elements of V are called vertices. The elements of E are called edges. The edge $e = \{u, v\} \in E$ is frequently abbreviated as either e or uv, whichever is more convenient for the discussion. Note that the edge uv is the same as the edge vu. As an example, define the graph G as $G = (V, E)$ where $V = \{a, b, c, d\}$ and $E = \{ab, bc, cd, da, bd\}$. The following figure gives a representation of the graph G and will be used to illustrate many of the upcoming definitions.

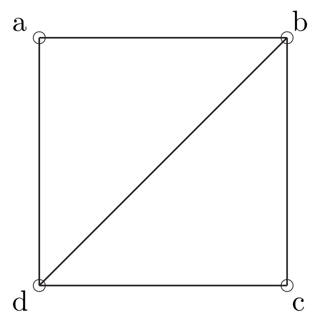


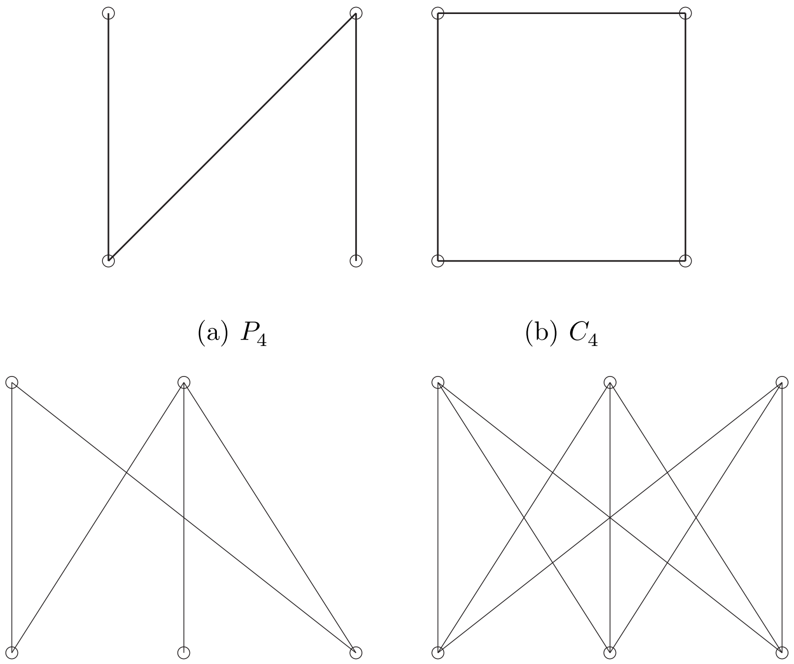
Fig. 2.1 G, an example of a graph

If $e = uv$ is an edge of G then u and v are said to be adjacent vertices and that e joins u and v. The vertex u and edge e are said to be incident with each other, as are v and e. Two edges e and f are adjacent if they are incident with a common vertex. To demonstrate this, in Fig 1.1 a is adjacent to b, ab joins a and b, a is incident to ab, and ab is adjacent to bc.

The neighborhood of a vertex v in the graph G, denoted $N_G(v)$, is the set $N_G(v) = \{u \in V : uv \in E\}$. The members of $N_G(v)$ are called the neighbors of v. The closed neighborhood of a vertex v, denoted $N_G[v]$, is the set $N_G[v] = N_G(v) \cup \{v\}$. If the graph under discussion is clear then we can omit the subscript and simply write the neighborhood and closed neighborhood of a vertex v as $N(v)$ and $N[v]$ respectively. If $S \subseteq V$, then the neighborhood of S is the set $N_G(S) = \bigcup_{s\in S} N(s)$ and the closed neighborhood of S is the set $N_G[S] = \bigcup_{s\in S} N[s]$. The degree of a vertex v in G, denoted $\deg_G(v)$, is defined as $\deg_G(v) = |N_G(v)|$. We can omit the subscript if there is no ambiguity of which graph is under discussion. Define the maximum and minimum degrees of the graph G to be $\Delta(G) = \max\{\deg(v) : v \in V\}$ and $\delta(G) = \min\{\deg(v) : v \in V\}$ respectively. If all the vertices of G have the same degree r, then G is said to be $r-$regular. A $3-$regular graph is called cubic. Referring back to Fig. 1.1, $N_G(a) = \{b, d\}$ and $N_G[a] = \{a, b, d\}$ where as $N_G(b) = \{a, c, d\}$ and $N_G[b] = \{a, b, c, d\}$. So $\deg_G(a) = 2$ and $\deg_G(b) = 3$. It can also be seen that $\Delta(G) = 3$ and $\delta(G) = 2$.

The complete graph with n vertices K_n has each vertex adjacent to every other vertex. Thus K_n has $\binom{n}{2}$ edges and is $(n-1)-$regular. A bipartite graph G is a graph whose vertex set V can be partition into two subsets V_1 and V_2 so that each edge of G has an end vertex in each of V_1 and V_2. If each vertex of V_1 is adjacent to each vertex in V_2 where V_1 has m vertices and V_2 has n vertices, then G is a complete bipartite graph, denoted $K_{m,n}$. A $u-v$ path of order n, denoted P_n, is a sequence of distinct vertices $P_n : u = v_1, \ldots, v_n = v$ beginning with u and ending with v such that $v_iv_{i+1} \in E$, for

$i = 1, 2, \ldots, n-1$. If we add the edge $v_n v_1$ to the path P_n then we call the resulting graph a cycle of order n, denoted C_n. If for each pair of vertices, u and v of G, there is a $u - v$ path, then G is said to be connected. A maximal connected subgraph of G is called a component. A tree is a connected graph with no cycles.

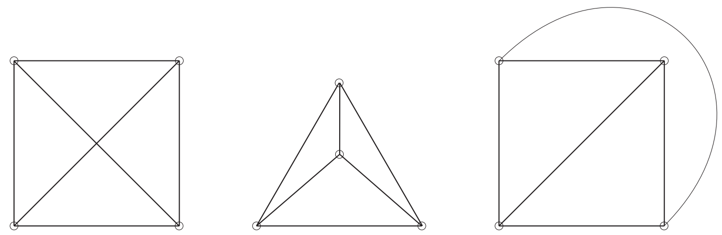


(a) P_4 (b) C_4

(c) bipartite graph (d) $K_{3,3}$

Fig. 2.2 Interesting Families of Graphs

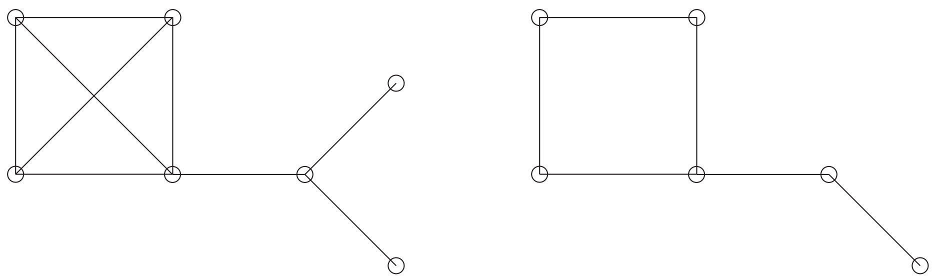
Two graphs G and H are isomorphic, denoted $G \cong H$, if there is a labeling of the vertices in such a way that there is a bijection $\varphi : V(G) \to V(H)$ that preserves adjacency; that is, $uv \in E(G)$ if and only if $\varphi(u)\varphi(v) \in E(H)$. Intuitively, two graphs are isomorphic if the vertices of one graph can be moved and the edges stretched in such a way that the two graphs look identical.



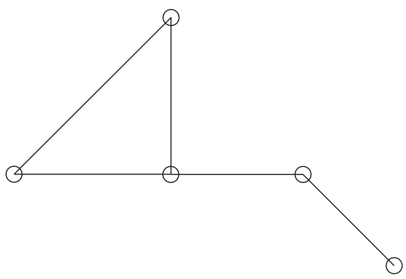
(a) K_4 (b) K_4 (c) K_4

Fig. 2.3 Three isomorphic K_4 graphs

A graph $H = (V(H), E(H))$ is a subgraph of the graph $G = (V(G), E(G))$, denoted $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The subgraph $H \subseteq G$ is an induced subgraph of G, denoted $G[V(H)]$, provided if $u, v \in V(H)$ and $uv \in E(G)$, then $uv \in E(H)$.



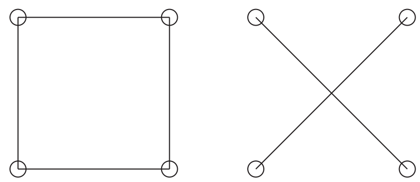
(a) Graph G (b) Subgraph of G



(c) Induced Subgraph of G

Fig. 2.4 A subgraph and an induced subgraph

The complement of a graph $G = (V, E)$, denoted $\overline{G}$, is the graph $\overline{G} = (V, \overline{E})$ where $uv \in \overline{E}$ if and only if $uv \notin E$.

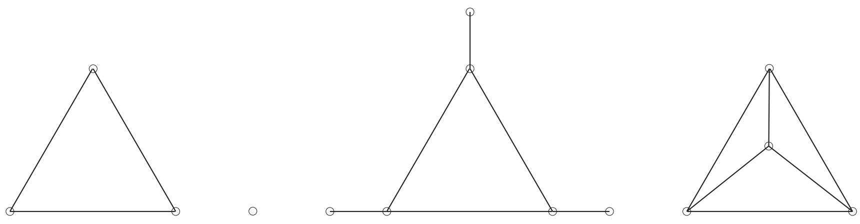


(a) Graph G (b) $\overline{G}$

Fig. 2.5 A graph and its complement

The union of two graphs G and H is the graph $G \cup H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. If the graphs G and H are disjoint, then the union is said to be a disjoint union and denoted $G + H$. To denote the graph of n copies of a graph G, the notation nG is used to describe the graph.

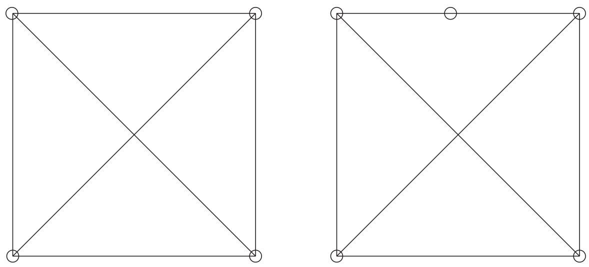
Let G be a graph with n vertices and H be a graph. The corona $G \odot H$ of G and H is the graph obtained by taking G with n copies of H and joining the ith vertex of G to every vertex of the ith copy of H. The following figure illustrates the corona operation and also shows that the corona is not a commutative operation.



(a) K_3 (b) K_1 (c) $K_3 \odot K_1$ (d) $K_1 \odot K_3$

Fig. 2.6 The Coronas of K_3 and K_1

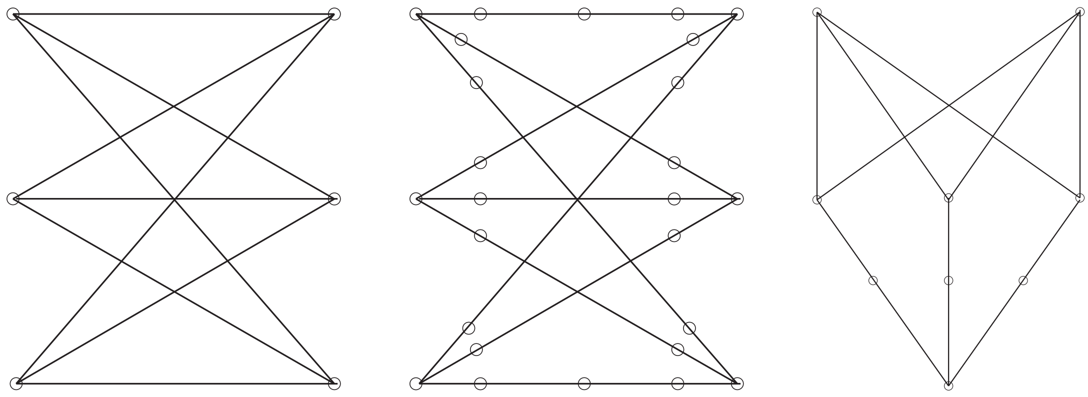
A subdivision of an edge uv in G gives a new graph G' where $V(G') = V(G) \cup \{w\}$ and $E(G') = E(G) \setminus \{uv\} \cup \{uw, vw\}$. A subdivision of a graph is obtained by a finite sequence of subdivisions of edges of the graph. Below we illustrate the idea of subdivisions of graphs.



(a) K_4 (b) Subdivision of K_4

Fig. 2.7 A Subdivision of K_4

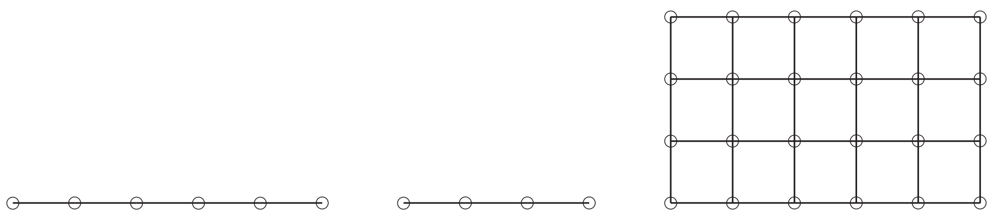
Two graphs G and H are homeomorphic if there exists a subdivision of G and and a subdivision of H so that the subdivision of G is isomorphic to the subdivision of H. Intuitively, the graph G is homeomorphic to the graph H if the graphs are homeomorphic in the topological sense. We give examples of homeomorphic graphs below.



(a) $K_{3,3}$ (b) Homeomorphic to $K_{3,3}$ (c) Homeomorphic to $K_{3,3}$

Fig. 2.8 Graphs Homeomorphic to $K_{3,3}$

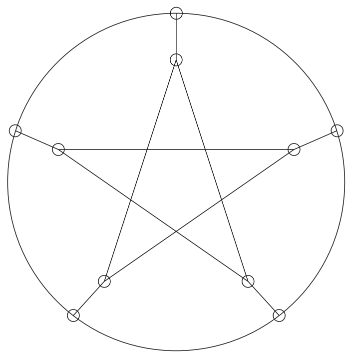
Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ each be a graph. The Cartesian product of G_1 and G_2, denoted $G_1 \times G_2$, is the graph $G = (V, E)$ where $V = V_1 \times V_2$ (i.e. the Cartesian product of the vertex sets), and the vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent if $u_1 = v_1$ and $u_2v_2 \in E_2$ or if $u_1v_1 \in E_1$ and $u_2 = v_2$. Note that the cartesian product is a commutative operation on graphs.



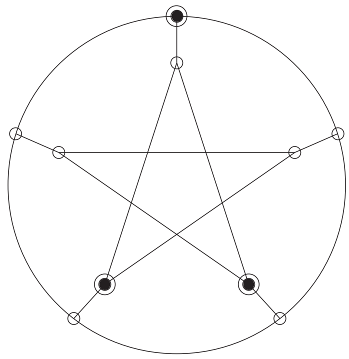
(a) P_6 (b) P_4 (c) $P_6 \times P_4$

Fig. 2.9 The Cartesian Product of P_6 and P_4

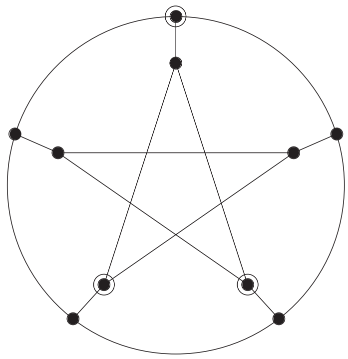
In a graph G, a set $S \subseteq V(G)$ is a dominating set of G if $N[S] = V(G)$. A vertex $v \in N[S]$ is said to be dominated by S. A minimal dominating set is a dominating set of minimum cardinality. The cardinality of a minimal dominating set is the domination number of G, denoted $\gamma(G)$. To show this graphically, the circled dark vertices will be the vertices in the dominating set and the dark vertices will be the vertices dominated by the dominating set.



(a) Petersen Graph



(b) Dominating Set



(c) The Petersen Graph is dominated

Fig. 2.10 An Example of Domination in Graphs

Chapter 3

Power Domination

An electric power network consists of electrical nodes (loads and generators) and transmission lines joining the electrical nodes. Electric power companies need to monitor the state of their networks continually. The state of the network is defined by a set of variables: the voltage magnitude at loads and the machine phase angle at generators [1]. One method of monitoring these variables is to place Phase Measurement Units (PMUs) at selected locations in the system. Because of the high cost of a PMU, it is important to minimize the number of PMUs used while still maintaining the ability of monitoring the entire system.

This problem was first studied in terms of graphs by T.W. Haynes, S.M. Hedetniemi, S.T. Hedetniemi and M.A. Henning in 2002 [8]. Indeed, an electric power network can be modeled by a graph where the vertices represent the electric nodes and the edges are associated with the transmission lines joining two electrical nodes. In this model, the power domination problem in graphs consists of finding a minimal set of vertices from where the entire graph can be observed according to certain rules. In terms of the physical network, those vertices will provide the locations where the PMUs should be placed in order to monitor the entire graph at the minimal cost.

A PMU measures the voltage and phase angle at the vertex where it is located, but also at other vertices or edges, according with the following propagation rules:

1. Any vertex that is incident to an observed edge is observed.

2. Any edge joining two observed vertices is observed.

3. If a vertex is incident to a total of k edges, $k > 1$, and if $k - 1$ of these edges are observed, then all k of these edges are observed.

Note that we followed the rules as presented in [8]. In [4] the authors present the propagation rules in a different way, that ultimately, as observed in [5], is equivalent to those above.

Algorithmically, given a graph $G = (V, E)$ and set of vertices $P \subset V$, we are going to construct a set of vertices C that can be observed from P and a set of edges F that are observed by P [5].

1. Initialize $C = P$ and $F = \{e \in E : e \text{ is incident to a vertex in } P\}$.

2. Add to C any vertex in $V - C$ which is incident to an edge in F.

3. Add to F any edge e in $E - F$ which satisfies one of the following conditions:

   a) both end-vertices of e are in C.

   b) e is incident to a vertex v of degree greater than one, for which all the other edges incident to v are already in F.

4. If steps 2 and 3 fail to locate any new edges or vertices for inclusion, stop. Otherwise, go to step 2.

The final state of the sets C and F give the set of vertices and edges observed by the set P. The power domination problem for a given graph G consists of finding a minimal power dominating set (PDS) for G. The cardinality of a minimal PDS in G is called the power domination number of G, and it is denoted as $\gamma_P(G)$. A power dominating set of G with cardinality $\gamma_P(G)$ is sometimes referred to as a $\gamma_P - set$.

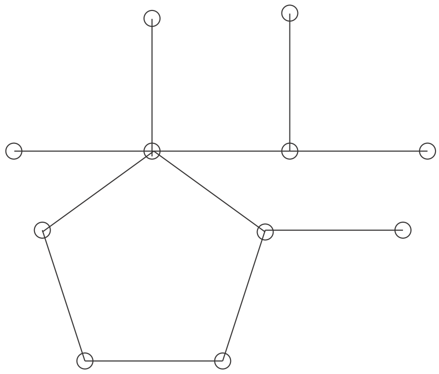
The problem of finding $\gamma_P(G)$ for a given graph G has been proven to be NP-complete even when reduced to certain classes of graphs, such as bipartite graphs and chordal graphs [8], or even split graphs [10], a subclass of chordal graphs. However, Liao and Lee [10] presented a linear time algorithm for finding the PDS of interval graphs, if the interval ordering of the graph is provided. If the interval order is not given, they provided an algorithm of $O(nlogn)$ and proved that it is asymptotically optimal. Other efficient algorithms have been presented for trees [9] and more generally, for graphs with bounded treewidth [9]. On block graphs [13] and claw-free graphs [14] there exists upper bounds given for the power domination number.

3.1 Structural Properties Related to γ_P

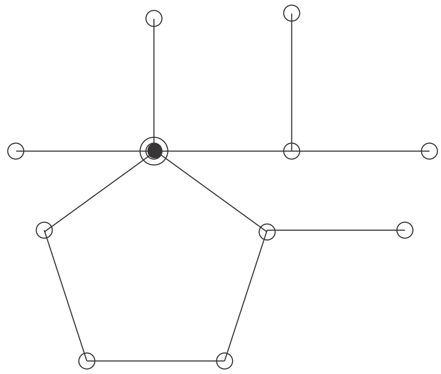
We begin by noting that there is a more simplified rule for propagation that is equivalent to those that are presented above. The equivalent propagation rule is the iterated process:

- If a vertex has a total of k neighbors, $k > 1$, and if $k - 1$ of these neighbors are observed, then all k of these neighbors are observed.

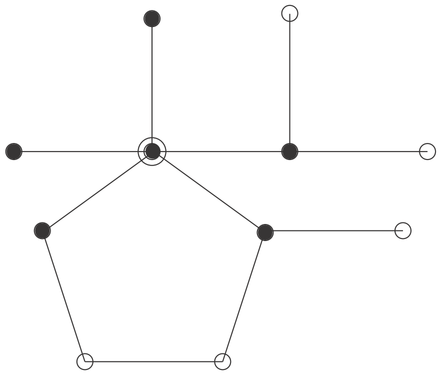
That is, if PMUs are placed on the vertices in $S \subset V(G)$ for a graph G, then $N[S]$ is observed. Next if v is observed and $deg(v) = k$ with $k - 1$ neighbors of v being observed, then all of the neighbors of v are observed. We repeat this process until no new vertices are observed, and the resulting set of observed vertices is the set of vertices that S observes. A demonstration of the propagation rules will now be given. The transparent vertices are unobserved while the darkened vertices are observed. The darkened vertices that are circled are vertices in the set where the PMUs have been placed.



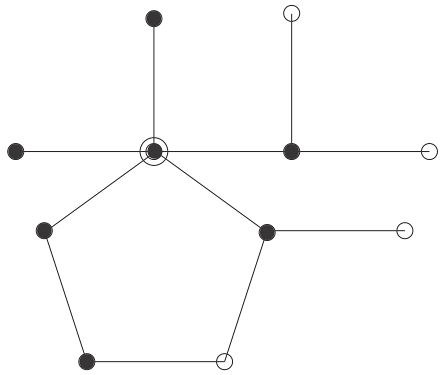
(a) Graph G



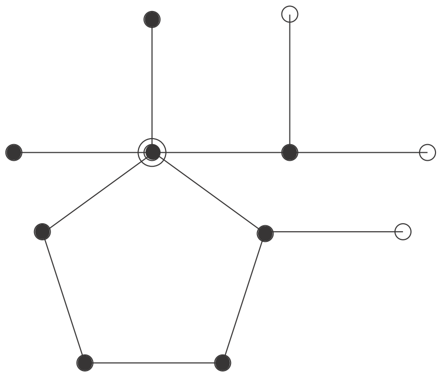
(b) PMU is placed on a vertex



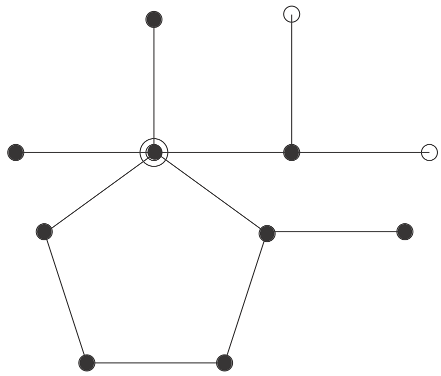
(c) Neighborhood is observed



(d) Propagation Occurs



(e) Propagation Occurs



(f) End of Propagation

Fig. 3.1 Propagation Rules Illustrated

In [8], the authors present many results on power domination and some will be utilized in showing new results.

Theorem 3.1.1. [8] *For any graph G, $1 \leq \gamma_P(G) \leq \gamma(G)$.* $\square$

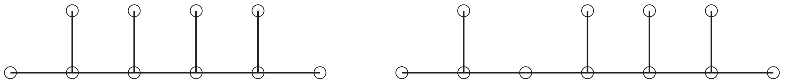
This bound should to be clear since any dominating set is also a power dominating set.

It was also pointed out in [8] that every graph H is the induced subgraph of a graph G satisfying $\gamma(G) = \gamma_P(G)$. The example that is considered in [8] is $G = H \odot \overline{K_2}$, which has H as an induced subgraph and $\gamma(G) = \gamma_P(G)$. The authors of [8] also point out that the difference $\gamma(G) - \gamma_P(G)$ can be arbitrarily large. To show this, we let $G = K_{1,k} \odot K_1$. Then $\gamma_P(G) = 1 < k + 1 = \gamma(G)$ and so as $k \longrightarrow \infty$ we have $\gamma(G) - \gamma_P(G) \longrightarrow \infty$.

The next theorem is a nice result that allows us to restrict the vertices that will be considered when looking at a power dominating set of a graph.

Theorem 3.1.2. [8] *Let G be a graph with $\Delta(G) \geq 3$. Then there is a γ_P − set S in which each vertex in S has degree at least 3.* $\square$

It should be noted that it is not necessarily true that if G is homeomorphic to G', then $\gamma_P(G) = \gamma_P(G')$. To illustrate this, Fig 3.2 shows two graphs that are homeomorphic but have different power domination numbers.



(a) $\gamma_P(T) = 2$ (b) $\gamma_P(T') = 3$

Fig. 3.2 Homeomorphic Graphs With Different Power Domination Numbers

Trees are studied extensively in [8] and the authors characterize all trees T satisfying $\gamma_P(T) = \gamma(T)$.

3.2 Known Bounds and Equalities for γ_P

In this section, known results for the power domination number for some families of graphs will be discussed. In [8], the authors present many observations on the power domination number for common families of graphs and we recall some of them here.

Theorem 3.2.1. *For the graph G where $G \in \{K_n, C_n, P_n, K_{1,n}, K_{2,n}\}$, $\gamma_P(G) = 1$.* $\square$

All trees T with $\gamma_P(T) = 1$ have been characterized.

Theorem 3.2.2. *For any tree T, $\gamma_P(T) = 1$ if and only if T is homeomorphic to the graph $K_{1,n}$ for some positive integer n.* $\square$

As noted earlier, the authors of [8] studied the power domination numbers of trees. For a tree T, define the spider number of T, denoted $sp(T)$, to be the minimum number of subsets $V(T)$ can be partitioned so that each subset induces a graph homeomorphic to $K_{1,k}$ for some $k \in \mathbb{N}$ (graphs homeomorphic to $K_{1,k}$ are sometimes called spiders).

Theorem 3.2.3. *For any tree T, $sp(T) = \gamma_P(T)$.* $\square$

The following sharp upper bound for the power domination number of a graph was initially shown for trees in [8] and was later generalized to all graphs in [14]. Define $\mathcal{T}$ to be the family of graphs obtained from connected graphs H by adding two new vertices v' and v'' to each vertex v of H and new edges vv' and vv''.

Theorem 3.2.4. *If G is a connected graph of order $n \geq 3$, then $\gamma_P(G) \leq \frac{n}{3}$ with equality if and only if $G \in \mathcal{T} \cup \{K_{3,3}\}$.* $\square$

An elusive problem in the domination problem in graphs is finding a closed formula for the domination number of a grid graph, $P_n \times P_m$ for some $m, n \in \mathbb{N}$.

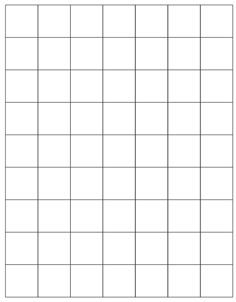


Fig. 3.3 The 8×10 grid, $P_8 \times P_{10}$

In [5], a surprising result was found. A closed formula for the power domination number of a grid graph $P_n \times P_m$ was found. What makes this surprising is that the (regular) domination problem for grids is still an open problem.

Theorem [5] If G is an $n \times m$ grid graph, $m \geq n \geq 1$ then

$$\gamma_P(G) = \begin{cases} \lceil \frac{n+1}{4} \rceil & , \text{if } m \equiv 4 \bmod 8 \\ \lceil \frac{n}{4} \rceil & , \text{otherwise.} \end{cases} \qquad \square$$

The power domination numbers for cylinders $P_n \times C_m$ for integers $n \geq 2, m \geq 3$, tori $C_n \times C_m$ for integers $n, m \geq 3$, and generalized Petersen graphs have also been studied. Tight upper bounds have been given for the power domination numbers of each of these families of graphs.

Theorem 3.2.5. [2] *The power domination number for the cylinder* $G = P_n \times C_m$ *is*

$$\gamma_P(G) \leq \begin{cases} \min\{\lceil \frac{m+1}{4} \rceil, \lceil \frac{n+1}{2} \rceil\} & , \textit{if } n \equiv 4 \textit{ mod 8} \\ \min\{\lceil \frac{m}{4} \rceil, \lceil \frac{n+1}{2} \rceil\} & , \textit{otherwise.} \end{cases} \qquad \square$$

The following corollary establishes the tightness of the bound given in the above result.

Corollary 3.2.6. [2] *The power domination number for the cylinder* $P_n \times C_m$ *for* $n \geq 2$ *is*

$$\gamma_P(P_n \times C_m) = \begin{cases} 2 & \textit{if } n = 2, 3 \textit{ and } 4 \leq m \\ 2 & \textit{if } 4 \leq n \textit{ and } 4 \leq m \leq 8. \end{cases} \qquad \square$$

Theorem 3.2.7. [2] *The power domination number for the torus* $G = C_n \times C_m$, $n \leq m$, *is*

$$\gamma_P(G) \leq \begin{cases} \lceil \frac{n}{2} \rceil & \textit{if } n \equiv 0 \textit{ mod 4} \\ \lceil \frac{n+1}{2} \rceil & \textit{otherwise.} \end{cases} \qquad \square$$

The discussion on the generalized Petersen graphs will be saved for the next chapter.

Chapter 4

The Generalized Petersen Graph $\gamma_P(P(m,k))$

To begin, we recall the definition of a Generalized Petersen graph. For $m \geq 3$, $k < m$, $k \geq 1$, and $\gcd(m,k) = 1$, the Generalized Petersen graph $P(m,k)$ is the graph with vertex set $\{v_0, v_1, \ldots, v_{m-1}\} \cup \{w_0, w_1, \ldots, w_{m-1}\}$ and edges $\{v_i w_i\}, \{v_i v_{i+1}\}, \{w_i w_{i+k}\}$ for every $i = 0, 1, \ldots, m-1$, where the subscript sum is in modulo m. The v_i vertices will be referred to as the "outside" vertices and the w_i vertices will be referred to as "inside" vertices. The $v_i w_i$ edges will be referred to as "spokes". The following figure is an example of a Generalized Petersen graph.

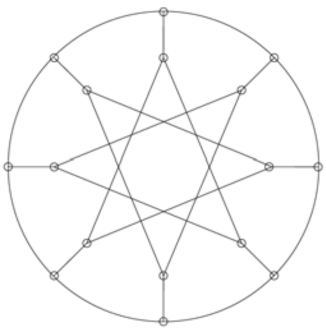


Fig. 4.1 The Generalized Petersen Graph $P(8,3)$

There are indeed positive integers k and l such that $P(m,k) \cong P(m,l)$. It should be clear that if $l = m-k$ then $P(m,k) \cong P(m,l)$. There are further characterizations of isomorphic generalized Petersen graphs. The following was first shown by Watkins [12] and then by Steimle and Staton [11].

Theorem [11; 12] Let $m > 3$ and $\gcd(m,k) = 1$, $\gcd(m,l) = 1$, and $kl \equiv 1 \bmod m$. Then $P(m,k) \cong P(m,l)$. $\square$

We will recall the isomorphism function used in the proof of the above Theorem. Label $P(m,k)$ with the outer vertices as $\{v_i\}_{i=0}^{m-1}$ and the inner cycle $\{w_i\}_{i=0}^{m-1}$ with v_i adjacent to v_{i+1}, w_i adjacent to w_{i+k}, and v_i adjacent to w_i for all i and subscripts in modulo m. Label $P(m,l)$ with the outer vertices as $\{x_i\}_{i=0}^{m-1}$ and the inner cycle $\{y_i\}_{i=0}^{m-1}$ with x_i adjacent to x_{i+1}, y_i adjacent to y_{i+k}, and x_i adjacent to y_i for all i and subscripts in modulo m. Define the function $\varphi : V(P(m,k)) \rightarrow V(P(m,l))$ by $\varphi(v_i) = y_{1+(i-1)l}$ and $\varphi(w_i) = x_{1+(i-1)l}$. Then φ is the desired isomorphism. The isomorphism classes of generalized Petersen graphs are further characterized in [11].

Theorem [11] Let $m \geq 5$. Let $\gcd(k,m) = 1$ and $\gcd(l,m) = 1$, and $2 \leq k, l \leq m-2$. If $P(m,k) \cong P(m,l)$, then either $l \equiv \pm k \bmod m$ or $kl \equiv \pm 1 \bmod m$. $\square$

In [11], the authors continue to show that the number of isomorphism classes of $P(m,k)$ is $\frac{\varphi(m)+\kappa}{4}$ where φ is the Euler phi-function and κ is the number of solutions to $x^2 \equiv \pm 1 \bmod m$. The following theorem is the known tight upper bound for the power domination number for the generalized Petersen graph.

Theorem 4.0.8. *The power domination number for the generalized Petersen graph $P(m,k)$ is bounded above by $\gamma_P(P(m,k)) \leq l'$ where $l' = min\{l : P(m,k) \cong P(m,l)\}$.* $\square$

To improve the known bound, the next statement is believed to be true but this still remains to be shown.

Conjecture 4.0.9. *If the vertices* $\{v_0, v_1, \ldots, v_k, v_{k+1}, v_{k+2}\}$ *are observed, then the graph* $P(m,k)$ *is observed.*

If the above conjecture holds, it would then imply the following improvement on the power domination number for a generalized Petersen graph.

Conjecture 4.0.10. *The power domination number for the generalized Petersen graph* $P(m,k)$ *is bounded above by* $\gamma_P(P(m,k)) \leq \lceil \frac{l'}{3} \rceil + 1$ *where* $l' = min\{l : P(m,k) \cong P(m,l)\}$.

Chapter 5

The Corona and the Power Domination Number

Recall that the corona of G and H, $G \odot H$, is the graph that results when we take one copy of G and $|V(G)|$ copies of H and connect every vertex in the ith copy of H to the ith vertex in G.

In this chapter some properties that relate the power domination number of a graph to the corona of graphs will be given. Earlier it was mentioned that every graph H is the induced subgraph of a graph G satisfying $\gamma(G) = \gamma_P(G)$ [8]. The family of graphs that were presented in [8] as an example of graphs where this equality is satisfied were $G = H \odot \overline{K_2}$. We present a similar result with a graph G and $G \odot K_1$.

Theorem 5.0.11. *Let G be a graph. Then $\gamma_P(G \odot K_1) = \gamma(G)$.*

Proof: Note that $\deg(v) \geq 2$ for each $v \in V(G)$ of $G \odot K_1$.

First, let us prove $\gamma_P(G \odot K_1) \leq \gamma(G)$. Let S be a dominating set for the graph G. Then $N[S] = V(G)$.

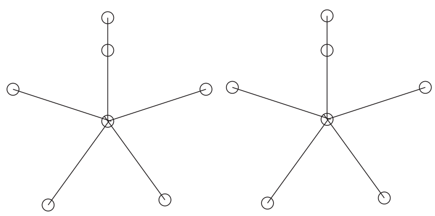
Now, let us prove $\gamma_P(G \odot K_1) \geq \gamma(G)$. Let S be a power dominating set for the graph $G \odot K_1$ with $|S| < \gamma(G)$. We can let each $s \in S$ be such that $s \in V(G)$ since $\deg(v) > 2$ only if $v \in V(G)$. Since $|S| < \gamma(G)$, there is some $u \in N(S)$ such that there is a $v \in G$ adjacent to u where $v \notin N(S)$. Furthermore, u has one more unobserved neighbor, its "spike". Thus, u has two unobserved neighbors and propagation does not occur. Since this holds for each such u, then there is some vertex $v \in G$ so that v is not observed in $G \odot K_1$. $\square$

The results that give relations between the corona of graphs and the power domination number depend largely on the fact that the degree of a vertex is increased, making propagation more difficult to occur in some instances. The idea of increasing the degree of a vertex by introducing a new edge in the graph gives the next result.

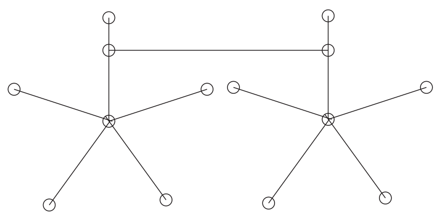
Corollary 5.0.12. *If $G = (V, E)$ is a graph then $\gamma_P(G \cup e) \leq \gamma_P(G) + 1$ where e is some edge not in G. Furthermore, this bound is tight.*

Proof: Let S be a $\gamma_P - set$ for G and let $e = uv$ for some $u, v \in V$. Then $S \cup u$ is a power dominating set as is $S \cup \{v\}$. Thus $\gamma_P(G \cup e) \leq \gamma_P(G) + 1$.

To show that this bound is tight, let the graph $G = K_{1,n}$ with one edge, say $e = uv$, subdivided exactly once into the edges ux, xv. Then $\gamma_P(2G) = 2$. But, $\gamma_P(2G \cup xx') = 3$. $\square$



(a) $\gamma_P = 2$



(b) $\gamma_P = 3$

Fig. 5.1 A Specific Case of the Class of Graphs Defined in Corollary 5.0.4

It is the hope that some investigation of Corollary 5.0.12 will lead to some further insight on the power domination problem. Perhaps the dynamics of randomly adding edges to a graph will reveal some information on the behavior of power domination.

Chapter 6

An Extension of Power Domination

In power domination, propagation occurs if all but one neighbor of an observed vertex are observed. A generalization of the propagation rules will be given, which will then be followed by some extensions of the known results for power domination.

6.1 $k-$Power Domination

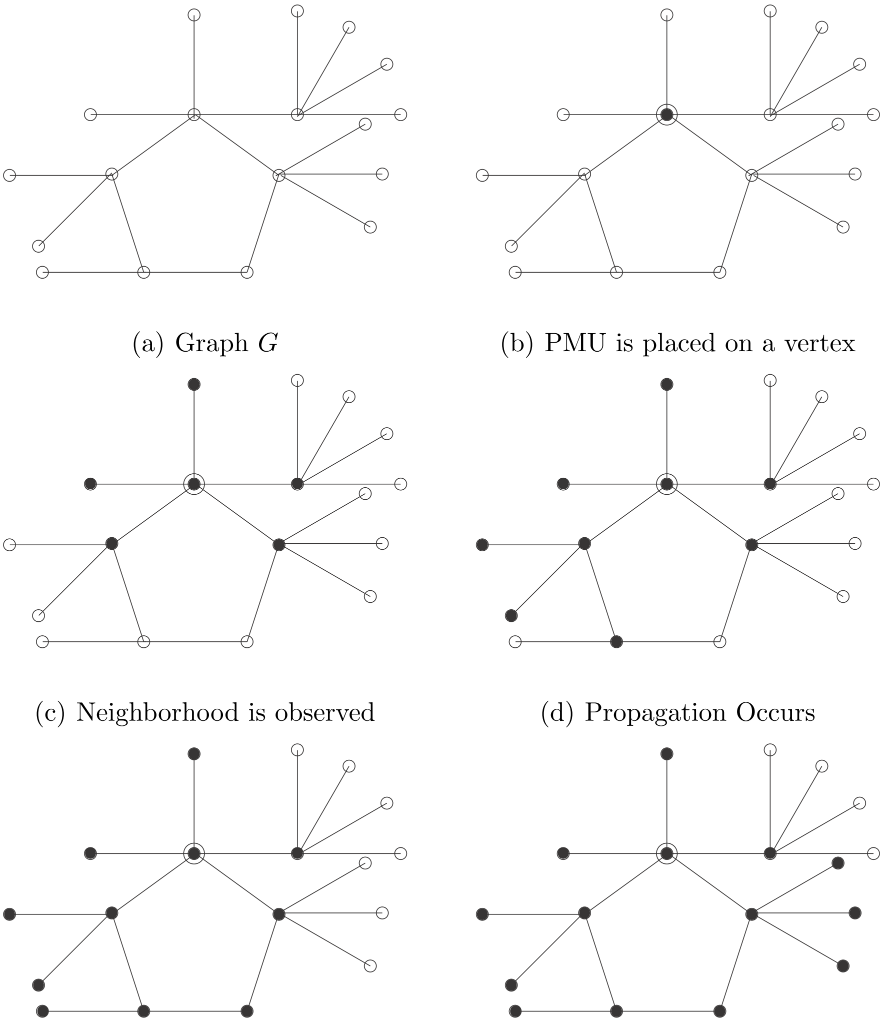
A generalization of power domination will be given in this section. In power domination, propagation occurs on an observed vertex v provided that all but one of the neighbors of v neighbors are observed. This comes from the fact that we can determine the unobserved neighbor's information by solving a system of equation. We forget that restriction and generalize power domination into $k-$power domination.

Let $S \subseteq V$ be a set. The vertices that are observed by S in $k-$power domination are:

1. $N[S]$ is observed.
2. Repeat the following until no new vertices are observed:
    (a) if $v \in V$ is observed with at most k neighbors of v begin unobserved, then $N[v]$ is observed.

Analogous definitions to those of power domination follow. For a graph $G = (V, E)$, if a set $S \subseteq V$ observes V then S is called a $k-$power dominating set. The size of a $k-$power dominating set of minimum cardinality will be denoted as $\gamma_k(G)$. A $k-$power dominating set of minimum cardinality may be referred to as a $\gamma_k - set$.

Notice that $k-$power domination only becomes interesting when $\Delta(G) \geq k + 2$. Indeed, power domination is the special case for when $k = 1$ and regular domination is the special case for when $k = 0$. The following results will not be too meaningful for the case when $k = 0$ though. To demonstrate a specific case of $k-$power domination, an illustration of the propagation rules will be given for when $k = 3$.



(a) Graph G

(b) PMU is placed on a vertex

(c) Neighborhood is observed

(d) Propagation Occurs

(e) Propagation Occurs

(f) End of Propagation

Fig. 6.1 Propagation Rules Illustrated for 3−Power Domination

For the remainder of this chapter, $k-$power domination will be under discussion.

6.2 Extensions of Results from Power Domination

An extension of Theorem 3.1.2 will be given. The proof is analogous to the proof of Theorem 3.1.2 that is presented in [8].

Theorem 6.2.1. *If G is a graph with $\Delta(G) \geq k+2$ for some integer $k \geq 1$, then G contains a $\gamma_k -$ set S where deg$(v) \geq k+2$ for each $v \in V$.*

Proof: Let S be a $\gamma_k - set$ and let $v \in S$ such that $\deg(v) < k+2$. Let u be a vertex with deg$\geq k+2$ with minimum distance from v. Then $S - \{v\} \cup \{u\}$ is a $k-$power dominating set. $\square$

This result is particularly nice for the same reason the analogous statement is in power domination. Theorem 6.2.1 gives a restriction on what vertices need to be considered as members of a γ_k-set. Like in power domination, this result will be used to show the following identity.

Theorem 6.2.2. *If G is a graph then $\gamma(G) = \gamma_k(G) \odot \overline{K_k}$.*

Proof: First let us prove $\gamma(G) \leq \gamma_k(G \odot \overline{K_k})$. Let S be a dominating set in G. Then S observes $\gamma_k(G \odot \overline{K_k})$.

Now let us prove $\gamma(G) \geq \gamma_k(G \odot \overline{K_k})$. Let S be a $k-$power dominating set for the graph $G \odot \overline{K_k}$ with $|S| < \gamma(G)$. We can let each $s \in S$ be such that $s \in V(G)$ since $\deg(v) > 2$ only if $v \in V(G)$. Let $u \in N(S)$. Then there is a $v \in G$ adjacent to u such that $v \notin N(S)$. Furthermore, u has k unobserved neighbors, namely the k "spikes". Thus u has $k+1$ neighbors unobserved. Since this holds for each such u, then there is some vertex $v \in G$ so that v is not observed in $G \odot \overline{K_k}$. $\square$

Along with Theorem 6.2.2, there are numerous other identities with power domination numbers one can make involving the corona operation. Some do not give any more information on what is going on in the graph or the power dominating sets. The identity presented above does give some insight about $k-$power domination when one wants to stop propagation.

The last result that will be extended is the upper bound for a general graph. Before continuing, recall that for a graph $G = (V, E)$, $\omega(G)$ is the number of connected components of G. If $X \subseteq V$, the $X-$private neighborhood of a vertex $v \in X$ is the set $pn(v, X) = N(v) \setminus N(X \setminus \{v\})$. External $X-$private neighbors are $X-$private neighbors not in X. The proof is analogous to the proof of Theorem 3.2.4 in [14].

Theorem 6.2.3. *If $G = (V, E)$ is a connected graph of order $n \geq k+2$ then $\gamma_k(G) \leq \frac{n}{k+2}$*

Proof: If $\Delta(G) \leq k + 1$, then $\gamma_k(G) = 1$.

Assume $\Delta(G) \geq k + 2$. Then G contains a γ_k-set where each vertex has degree at least $k + 2$. Let S be a γ_k-set so that $\omega(G[S])$ is minimum.

Claim: For each vertex $v \in S$, $|pn(v, S) \setminus S| \geq k + 1$.

Suppose otherwise. That is, suppose some vertex $v \in S$ is such that $|pn(v, S) \setminus S| \leq k$. If v is adjacent to some vertex in S, let $S' = S \setminus \{v\}$. Note that v and its neighbors neighbors except for possibly k are all observed by S'. These points are then observed by v. Hence, S' is a power dominating set of G and $|S'| < |S|$, a contradiction.

It must then be that v is an isolated vertex in $G[S]$. That is, $N(v) \subseteq V \setminus S$ and $|N(v) \setminus S| \geq k+2$. Choose $u \in N(v) \setminus pn(v, S)$ and let $S'' = (S \setminus \{v\}) \cup \{u\}$. Then v and its neighbors except for possibly k are observed by S''. These vertices, if they exist, are then observed by v. Thus S'' is a γ_k-set with $\omega(G[S'']) < \omega(G[S])$, which contradicts S being chosen with $\omega(G[S])$ being minimum.

It follows that G has at least $(k + 2)|S|$ vertices and so $\gamma_k(G) \leq \frac{n}{k+2}$. $\square$

Chapter 7

Conclusion

Finally, some open problems that may shed some light on the properties behind power domination will be discussed.

Graphs with square meshes have been studied for many families of graphs. Grids, cylinders, and tori are amongst those studied. The technique to observe each of these families of graphs is to arrange the vertices in the power dominating set in "knight moves" apart from each other. An analog may exist for triangular meshes and hexagonal meshes. Similar arguments to those of square meshes may be followed to find the power domination numbers of hexagonal or triangular grids, cylinders, and tori.

The best known bound for the power domination number of a graph G of order n is $\frac{n}{3}$. This bound depends on the number of vertices of G. There are many examples of arbitrarily large graphs (grids, cylinders, tori, etc.) with small power domination numbers. For example, we can have a grid $P_6 \times P_m$. We know that $\gamma_P(P_6 \times P_m) = 2$ for all integers $m \geq 5$. So we can have the same power dominating set for a graph with arbitrarily large order. The endeavor would be to find a better upper bound for general graphs that does not depend on the order of the graph.

The endeavor with exploring $k-$power domination is to hopefully shed some light on the power domination problem. What is meant by this is that by altering what vertices become observed through propagation, perhaps some property will shine through and reveal what really happens during power domination. Power domination does seem to be a way of studying the more general $k-$power domination though. As shown in the chapter on $k-$power domination, some of the structural properties of power domination are easily extended into $k-$power domination.

Another approach one may want to take is to randomly add edges to a graph and see how some invariants change with the addition of edges. This may lead to some property being revealed about what power domination depends on.

It was mentioned earlier on that if G and H are homeomorphic graphs, it is not necessarily true that $\gamma_P(G) = \gamma_P(H)$. It may be the case though, that there is a family of graphs $\mathcal{F}$ so that if G and H are homeomorphic and there is not some $F \in \mathcal{F}$ such that $F \subseteq G$ or $F \subseteq H$, then $\gamma_P(G) = \gamma_P(H)$. The problem then becomes finding the family of forbidden subgraphs $\mathcal{F}$ so that if there is no subgraph of two homeomorphic graphs, G and H, in $\mathcal{F}$, then $\gamma_P(G) = \gamma_P(H)$.

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